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This supplementary modern mathematics textbook is to help in-service teachers to broaden their background in mathematical concepts and ideas for grades 1 through 9. This in-service course was written with two basic objectives: (1) to help teachers to become familiar with some of the newer mathematical ideas and concepts for grades 1 through 9, and (2) to suggest ways in which creative learning can become part of the school program in mathematics. Materials from rectangular coordinates, the arithmetic of signed numbers, the use of variables, and the concept of mathematical function are developed in this text. (RP)

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MADISON PROJECT

SYRACUSE UNIVERSITY • WEBSTER COLLEGE

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SUPPLEMENTARY MODERN MATHEMATICS

FOR GRADES 1 THROUGH 9

IN-SERVICE COURSE FOR TEACHERS

FIRST COURSE: SIGNED NUMBERS, VARIABLES, FUNCTIONS,
AND CARTESIAN CO-ORDINATES

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Written portions of the present In-Service Course I have been prepared by Donald Cohen, Robert B. Davis, Roy Hajek, and Katharine Kharas. Filmed portions have been prepared by Beryl S. Cochran.

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This "Packaged" In-Service Course for Teachers is designed for an in-service course meeting once a week, for approximately 15 meetings. Each meeting is expected to last about 90 minutes.

Each meeting should be planned in advance by a person serving as "discussion leader," or by a group of people serving as a "steering committee."

We suggest that, at least for the first few meetings, this "package" should be adhered to as closely as possible.

We also suggest that the "Discussion Leader" is not the "teacher" for the course. Every participant shares the responsibility for building an intelligible course out of the "dehydrated" ingredients in this package. Each participant should contribute as much as possible. The responsibility of the "Discussion Leader" is mainly to organize these contributions.

Before the first session of the in-service course, the "Discussion Leader" or "Steering Committee" should prepare in advance by:

1. Reading all of the written material listed on the "Agenda" (that is, the "Introduction," "Signed Numbers," "Discussion of film excerpts," "Crossed Number Lines," and "Rules for Tic Tac Toe").

2. Viewing the four film excerpts ("Pebbles-in-the-Bag," "The Number Line," "Crossed Number Lines," and "Tic Tac Toe").

3. Making sure that he understands the "Pebbles-in-the-Bag" game, and the two versions of "Tic Tac Toe" that are contained in this lesson.

4. It is important to arrange good facilities for viewing the films. You need a fully darkened room, a good screen, and a high-quality 16mm. sound motion picture projector with a hi-fidelity sound system. The room must have good acoustics. You may also need extension cords, and spare bulbs for the projector.

SUPPLEMENTARY MODERN MATHEMATICS

For Grades 1 - 9

An In-Service Course for Teachers

FIRST SESSION

Agenda:

1. Introduction:
   - Broadening the K - 8 Program
   - Creativity
2. Signed Numbers: You try it, using the "pebbles-in-the-bag" model
3. Signed Numbers -- First Film Excerpt: "Pebbles-in-the-Bag"
4. Discussion of first film excerpt (some questions are provided if you care to use them)
5. The Number Line: You Try It Yourself!
6. The Number Line: Second Film Excerpt
7. Discussion of the Second Film Excerpt
8. Crossed Number Lines: You Try It Yourself!
9. Crossed Number Lines: Third Film Excerpt
10. Discussion of third film excerpt (some questions are provided if you care to use them)
11. Rules for two modified versions of Tic Tac Toe
12. Fourth Film Excerpt
13. You Play Tic Tac Toe -- use Version II ("4 in a line" required to win, on a 5 x 5 board; there are fewer ties this way.)

Before the Next Session:

1. Try the "pebbles-in-the-bag" idea with your own classes!
2. Please read the section "General Remarks and Background Information."

1. Introduction

Broadening the K-8 Program

In recent years it has come to be recognized that the "traditional" school mathematics program concerns itself with a very small portion of mathematics. A far larger part of mathematics -- equally easy to learn, and equally important to know -- is excluded. Over the next few decades we face the task of broadening the content of the school mathematics program. Ideas such as rectangular (or "Cartesian") coordinates, the arithmetic of signed numbers, the use of variables, the concept of mathematical function, and so on, are coming to be included, often in grades two through six, in most "modern" programs.

ONE PURPOSE OF THIS IN-SERVICE COURSE IS TO HELP TEACHERS TO BECOME FAMILIAR WITH SOME OF THESE "NEW" IDEAS.

Creativity

A second aspect of "modern" mathematics is a greater emphasis upon creativity. The past fifty years has seen an admirable development of creative learning experiences for children in painting, sculpture, writing poetry, writing plays, and so on. This creative approach is just as valid, and just as important, in mathematics and science as it is in art and poetry. A second purpose of this course is to suggest ways in which creative learning can become part of the school program in mathematics -- although this goal may sound strange indeed to anyone who has never encountered a "creative" approach in mathematics. Just what both of these purposes mean in actual practice will, we hope, become clear as one studies the materials which follow.

2. Signed Numbers: You try it, using the "pebbles-in-the-bag model.

BEFORE viewing the film, TRY THIS YOURSELF!

When we count, we use numbers like 1, 2, 3, 4, . . . and so on. (These are often called the "counting numbers.") We do not use "negative one," or "negative seven," or "negative one hundred," etc.

If we want to introduce a child to signed numbers (positive and negative), we need to use something other than simple counting.

Where do signed numbers make their appearance?

The answer is that signed numbers appear when we select an ARBITRARY REFERENCE POINT, from which we can depart in either of two directions.

Examples:

1. On a thermometer, we mark an arbitrary "ZERO." Temperatures may be "above zero" or else "below zero." If the usual language of mathematics were used, we would read "10 degrees above zero" as "positive ten," and "10 degrees below zero" as "negative ten."

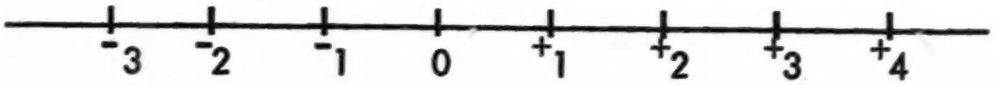
2. At a missile launching, everything is timed in relation to the instant of launching. This arbitrary reference time is labeled "ZERO" or "BLAST OFF."

The famous "counting backwards" which precedes this is not really backwards. Rather, we are counting

"negative three, negative two, negative one,

BLAST OFF, positive one (i.e., one second
after blast-off), positive two, . . . "

3. The most important "picture" of the real numbers is provided by the number line. Here is a portion of this picture, which can be extended further to left or right:



The "Pebbles-in-the-Bag" Model

In order to introduce the notion of signed numbers to the young child, we can create a situation in which their use is simple and natural. We can do this by means of the "pebbles-in-the-bag" model as follows:

1. We have a bag, containing a large number of pebbles. We do not know how many pebbles are in the bag at the outset of our game. We never use this number.

2. We have a large number of pebbles not in the bag. (Because of items 1 and 2, we can put pebbles into the bag, or take pebbles out of the bag, without running out of pebbles in either operation.)

3. We begin the game by having some child (or a participant in the in-service course) say "Go!" This establishes our arbitrary reference point.

4. Suppose Jim said "Go." If, as a result of putting pebbles into the bag, taking them out, putting them in, etc., we decide that we have 8 more pebbles in the bag than we had when we started, we express this as

"positive eight."

If we have, at some point, 3 less pebbles in the bag than we had when Jim said

"Go," we express this as

"negative three."

If we have the same number of pebbles in the bag as there were when Jim said "Go," we express this as

"zero."

5. "Positive eight" is written

$$^{+}8,$$

using a small sign, written above the middle of the line, so that we will not confuse it with the sign for addition.

"Negative three" is written, similarly, as

$$^{-}3.$$

We find in working with children that it avoids confusion if we use two different symbols (and two different words to read the symbols) to express two different mathematical concepts. Hence we use "+" (read "plus") only to indicate the operation of addition, and we use "$^{+}$" (read "positive") to indicate a positive number -- that is, a number that is more than zero.

Similarly, we use "−" (read "minus") to indicate the operation of subtraction, and we use "$^{-}$" read "negative" to indicate a negative number -- that is, a number less than zero.

Thus, it would be correct to write

$$3 - 5 = {}^{-}2,$$

and to read this as "three minus five equals negative two." Do you understand this distinction?

6. We determine the state of pebbles in the bag, relative to the arbitrary reference

level, by keeping track of how many pebbles we put into the bag, and how many we take out. We never count the total number of pebbles in the bag.

(If we reduced this to a "counting" problem, we would have no use for signed numbers. The model is intended to introduce the children to signed numbers.)

Before viewing the film, try this yourself:

7. Have some participant say "Go." (We'll call the participant "Jane.")

8. Put 4 stones in the bag.

9. Before you forget, write "4" on the blackboard.

10. Are there more stones in the bag than there were when Jane said "Go," or are there less?

11. How many more?

12. Take 2 stones out of the bag.

13. Before you forget, write on the board; you should now have on the board:

$$4 - 2 .$$

14. Are there more stones in the bag now than when Jane said "Go," or less?

15. How many more?

16. How do we write this?

17. You should now have on the board: $4 - 2 = {}^{+}2$.

18. How do we read this? (Cf. item 5, above.)

19. Start over. Have someone else say "Go." Repeat, using different numbers.

20. Repeat as often as you like. Each time, have someone else say "Go," so that you do not confuse the arbitrary reference points. You can make the answer come out positive, negative, or zero.

21. How would you read each of these? How would you act them out?

$$5 - 3 = {}^{+}2$$

$$3 - 5 = {}^{-}2$$

$$7 - 7 = 0$$

$$6 - 10 = {}^{-}4$$

YOU ARE NOW READY TO VIEW THE FIRST FILM EXCERPT.

4. Discussion of First Film Excerpt.

1. Information about the film. The excerpt shows part of a class of second graders from Weston, Connecticut, an upper-class suburb of New York. The lesson is essentially a first lesson, although the teacher did meet the children very briefly about a week before the filming. On this occasion, the teacher used the pebbles-in-the-bag story, so this part is not entirely new. The teacher found that, even on the brief earlier session, several children already knew the words "positive" and "negative," evidently from older brothers and sisters; they did not, however, use these words accurately (and they still do not in the first part of this excerpt you just viewed). Fairly often, if you ask a class "does anyone know how we show that it's 2 more?" some child will know; if not, the teacher might say, "well, we write it this way ($^+2$) and read it as "positive two" -- that means we have two more."

Even where the question is purely rhetorical, it usually serves to get the students' attention well focused on the task.

Here are some questions you may wish to discuss. Better still, discuss the questions that you, yourself, are curious about.

2. At what grade level would you introduce the "pebbles-in-the-bag" model for signed numbers?

3. Where does this topic (of signed numbers) lead?

4. Where, in the school program, can you use signed numbers?

5. If you wish, practice "acting out" the pebbles-in-the-bag game again, as you did before viewing the film.

6. What can go wrong when you try this in class?

5. The Number Line: You Try It Yourself!

Before viewing the second film excerpt, TRY THIS YOURSELF!

1. Can you draw an incomplete picture of the number line on the board? (It is not possible to draw a "complete" picture; one can always extend in either direction.)

2. Can you locate 0 on the number line? Can you locate $^{+}1$? $^{+}2$? $^{-}1$? $^{-}2$? Can you locate $2\frac{1}{2}$ on the number line?

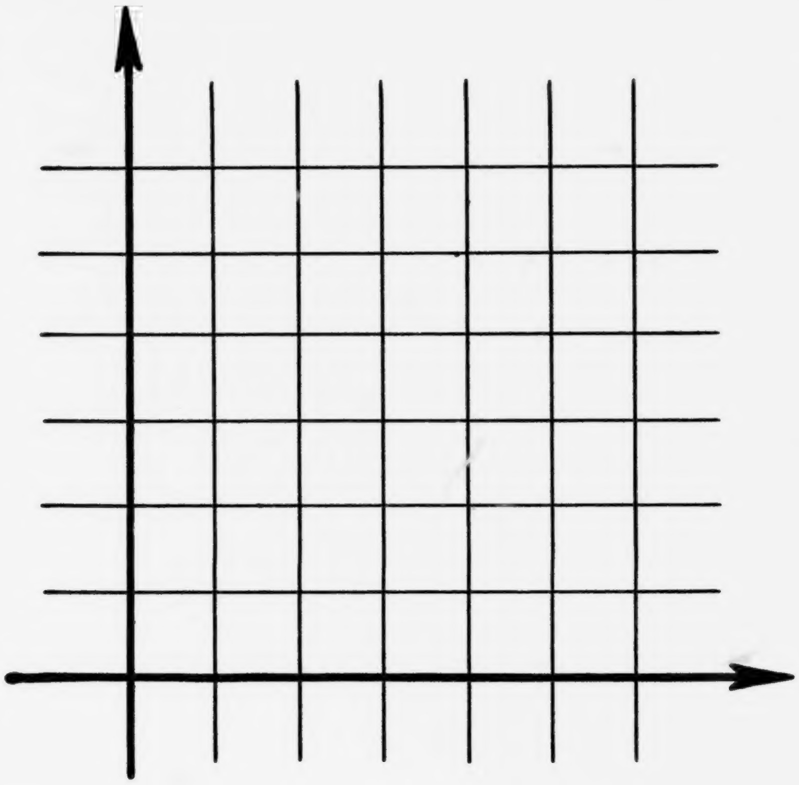
3. How do you use the number line in your own teaching?

4. Are you ready to view the second film excerpt?

7. Discussion of the second film excerpt

You may be interested to know that the basic idea behind most modern uses of the "number line" was introduced by Rene Descartes in the first part of the 17th century. This procedure of associating numbers with points on a line enabled Descartes to develop the subject known as analytic geometry, and virtually to unify algebra, arithmetic, and geometry into a single subject. This is the foundation for a great deal of modern work in geometry -- and, indeed, in nearly every branch of mathematics.

In next week's lesson we shall study Descartes' device of "crossing two number lines" so as to obtain Cartesian coordinates:



This will give us a mathematical tool that can be important in every grade, from kindergarten onward, and in a wide variety of subjects.

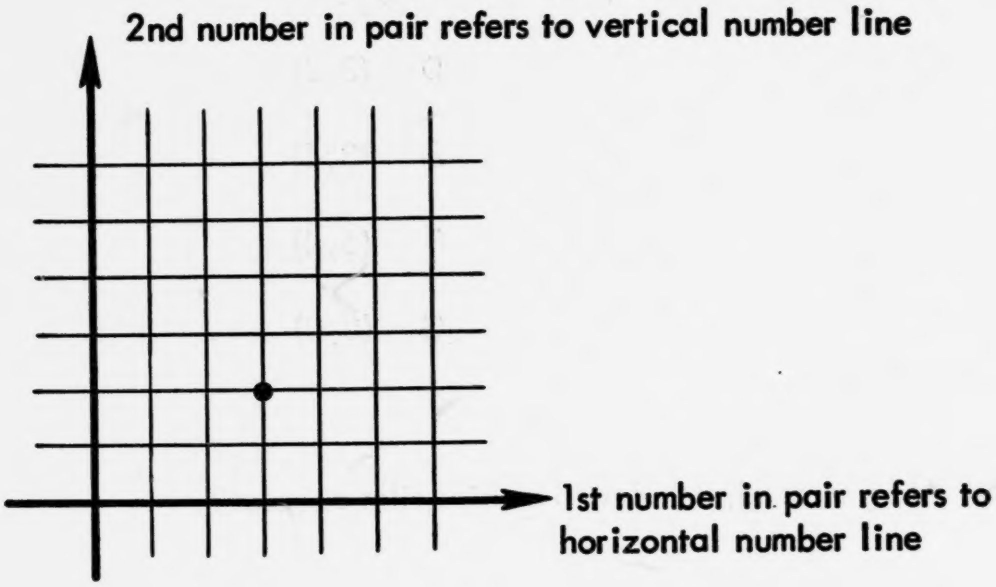
8. Crossed Number Lines: You Try It Yourself!

BEFORE viewing the third film excerpt, TRY THIS YOURSELF!

1. The number line gives us a valuable way to picture numbers. From another point of view, it lets us use numbers to describe points on a line.

2. By crossing two number lines, we can use an ordered pair of numbers to describe points in the plane. This invaluable device was first thought of by Rene Descartes (1596-1650), and is the foundation of much modern work in mathematics (e.g. what is variously called "Cartesian geometry," or "analytic geometry," or "co-ordinate geometry").

3. The point

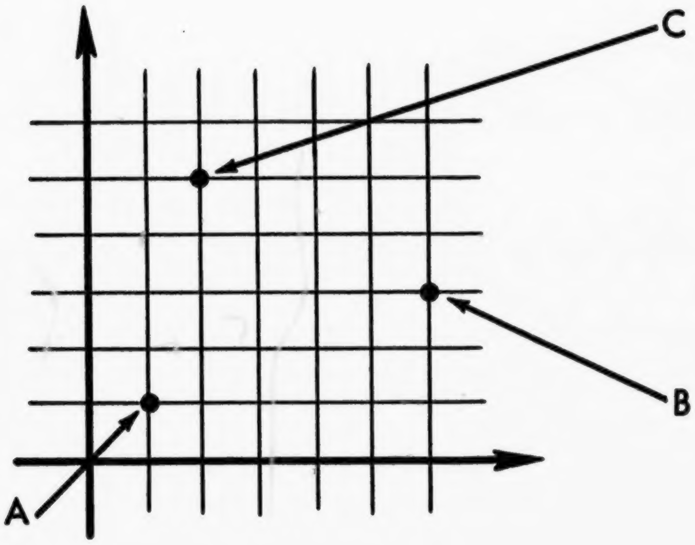

2nd number in pair refers to vertical number line
1st number in pair refers to horizontal number line

is called

(3,2).

We count from the heavy lines (known as "axes"), and we count "city blocks" rather than "intersections."

4. What numbers describe these points?



Answers:	A	(1,1)
	B	(6,3)
	C	(2,5)

5. Can you mark these points on the graph?

D	(3,2)
E	(3,6)
F	(5,4)
G	(0,0)

6. Are you ready to view the third film excerpt?

10. Discussion of third film excerpt

Here are some questions if you care to use them. Better still, use your own!

1. Where can you use graphs (or "crossed number lines") in the school program?

2. How can Cartesian co-ordinates ("crossed number lines") be used at various grade levels?

11. Rules for Two Modified Versions of Tic Tac Toe

Version I. (Shown in fourth film excerpt.)

1. Points are marked at intersections on a graph.

2. Size of graph: 5 by 5 [That is, the "corners" are at:

(0,0)
(4,0)
(0,4)
(4,4)]

3. Two teams play. One team marks its points X, the other ● .

4. You must locate a point by using the ordered pair of numbers to describe it. You may not mark it on the graph yourself. The referee marks points on the graph.

5. If the pair of numbers you call out corresponds to an illegal move, your team loses that turn.

6. To win, you must get 5 points (all belonging to the same team) in a straight line.

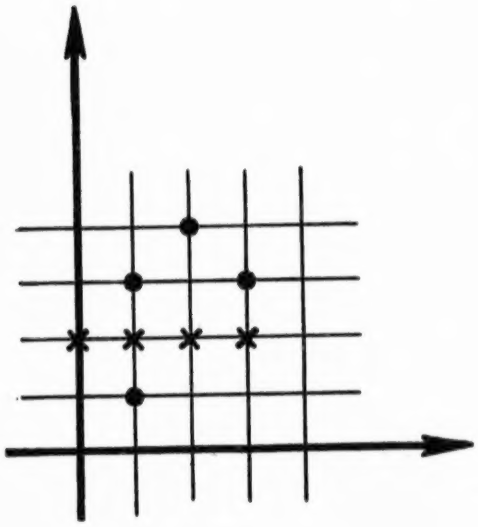
Version I is easy for defense, hard for offense. Consequently, it usually leads to a tie. As a result, we do not usually use it for long. Instead, we use Version II, below.

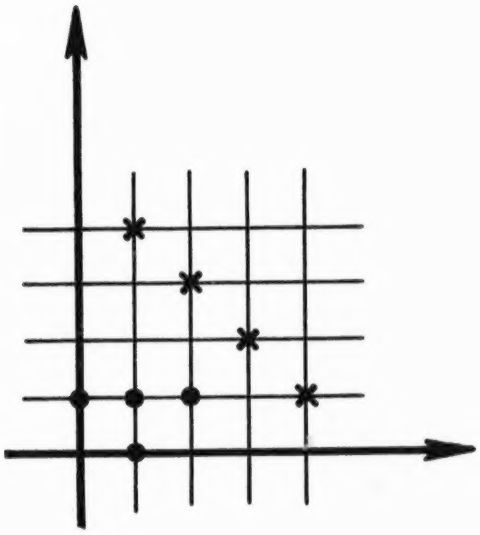
Version II. (Not shown in film.)

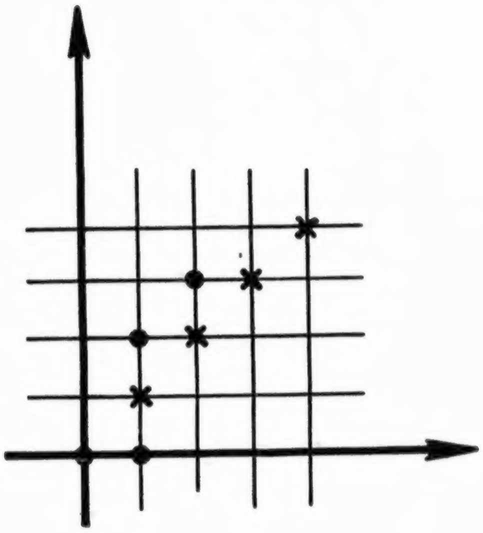
1. The rules 1-5 of Version I are used without change.

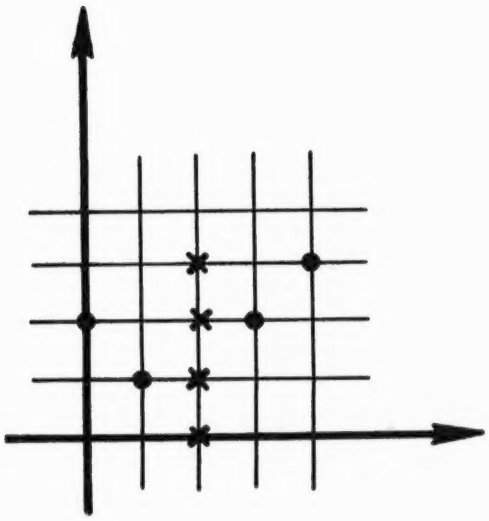
2. However, to win, you are required only to get four points (all belonging to your team) in an uninterrupted straight line.

Examples: In each case below, the X team has won:









Are you ready to view the fourth film excerpt?

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SUPPLEMENTARY MODERN MATHEMATICS

For Grades 1-9

REFERENCE BOOK

First Session

REFERENCE MATERIAL

for the First Session

The kind of questions that are likely to come up for discussion as you study this material do not admit of clear cut "answers." They're the kind of questions one needs to think about, and every now and then go back and think about again.

Nonetheless, if it will be of any use to you, here are some of our thoughts on these questions.

Agenda Item 2. Signed Numbers

Answers to question 21, page I-8

$5 - 3 = {}^{+}2$ might be read "five minus three equals positive two."

You could "act this out" by having a child say "Go!," then by putting 5 pebbles into the bag, and taking 3 pebbles out of the bag. There would then be 2 more pebbles in the bag than when the child said "Go."

$3 - 5 = {}^{-}2$ might be read "three minus five equals negative two."

You could act this out by having a child say "Go!," then putting 3 pebbles into the bag, and then taking 5 out. There would then be 2 less pebbles in the bag than when the child said "Go."

$7 - 7 = 0$ might be read "seven minus seven equals zero."

You could act this out by having a child say "Go!," then putting 7 pebbles into the bag, and then taking 7 out. There would then be neither more nor less pebbles in the bag than when the child said "Go!," but the same amount.

$6 - 10 = {}^{-}4$ might be read "six minus ten equals negative four."

You could act this out by having a child say "Go!," then putting 6 pebbles into the bag, and then taking 10 out. There would then be 4 less pebbles in the bag than when the child said "Go!"

Agenda Item 4. Discussion of first film excerpt

1. We ourselves have used the "pebbles-in-the-bag" model for signed numbers as an introduction at whatever point in the school program such an introduction was needed. This might be as early as the first grade, more likely around the second grade or later, and possibly as late as the 7th, 8th, or 9th grade. In general mathematics classes, this might even be at the 10th grade level. The important thing is to put it at the right point in your program, wherever the introduction of signed numbers is most advantageous. Considerable local variation here is inevitable.

2. The topic of signed numbers is extremely important as one gets into work in algebra and science, and can even be important for work in arithmetic. There are many possible tie-ins with other school work, for example: temperatures below zero are essentially expressed as negative numbers; altitude above and below sea level can be expressed using positive numbers for altitude above sea level, and negative numbers for altitude below sea level; dates A. D. and B. C. can be expressed in terms of positive and negative numbers, except that an error was made in the calendar, and the year zero was omitted. (The children may find this omission amusing.) Double entry bookkeeping, with credits and debits, can be expressed in signed numbers, with positive numbers representing being "in the black," and with negative numbers representing being "in the red."

In Session III we will be discussing an extremely interesting use of signed numbers--one which gives a possible answer to a question some of you may be wondering about now: what happens when I come to teach a subtraction problem like

$$\begin{array}{r} 32 \\ -\ 18 \\ \hline \end{array}$$

and say, "Since you can't take 8 from 2, we'll borrow ..."?

Evidently signed numbers appear quite naturally on the number line, when we begin to ask about extensions to the left. In a similar way, they appear in Cartesian co-ordinates, as we shall see within the next few lessons (where they are involved in work in the second, third, and fourth quadrants).

3. We have discussed this above, in answer to question 2.

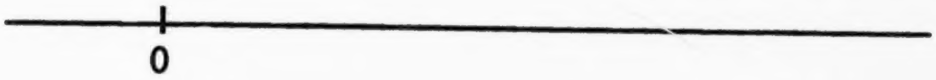
4. No comment is needed here, probably.

5. We have found in teaching pebbles-in-the-bag that the children get most easily confused when we either forget to have a child say "Go," or we forget to refer everything that happens afterward to the instant when he said "Go." If the reference point becomes obscure, chaos is likely -- the question "are there more than?" leads to the question "more than we have just now, or more than we had 5 minutes ago, or more than we had before we put all of those in, or just what?" This is, by the way, why we have _different_ children say "Go" each time -- if John says "Go" for two episodes, then the children can become confused as to which time is meant when one asks "are there more pebbles in the bag than when John said 'Go,' or are there less?"

Agenda Item 5. The Number Line

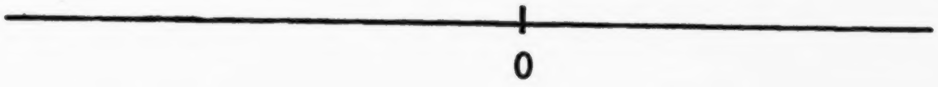
2. As you may have already discussed, we have freedom in where we choose to locate 0; for example, we might mark it like this:

a



or this:

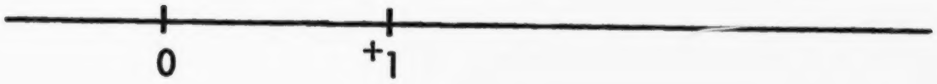
b



or in lots of other places.

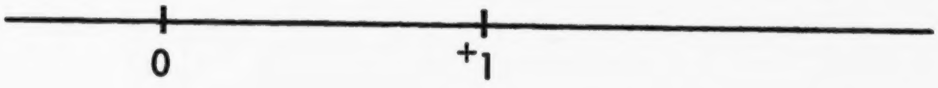
After marking 0, we still have freedom in our choice of location for $^+1$; we could do this for line a:

a_1



or this:

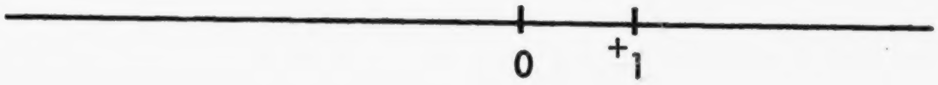
a_2



or lots of others.

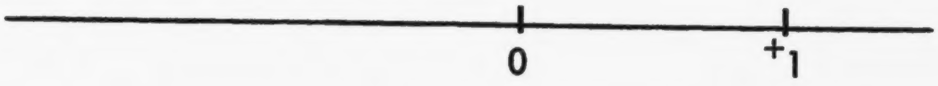
Similarly, for line b, we might do this:

b_1



or this:

b_2

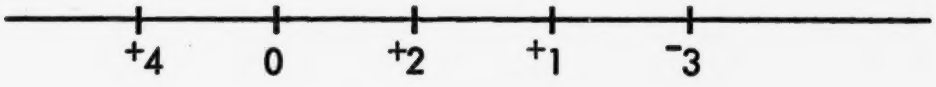


(There is, of course, no mathematical reason why $^+1$ is marked to the right of 0 -- it's just usually done that way unless a particular situation makes it useful to do otherwise.

It becomes particularly advantageous to have such a convention when we come to cross two number lines.)

After locating 0 and $^{+}1$ on the line, do we have a choice about where we put $^{+}2$, $^{-}1$, $^{-}2$, etc.? The answer is "no," if we assume that we want our number line to have two properties which make it a particularly useful model for children. One of these properties is that of order; for example, if we didn't care about order, we could locate numbers randomly like this:

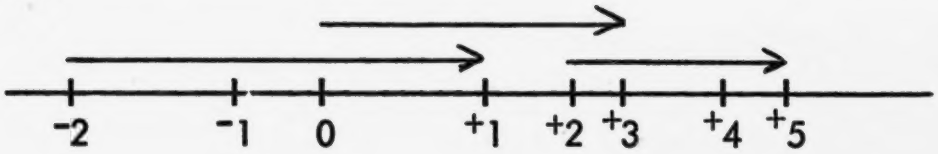
1.



(In this case, a child would not be able, for example, to think of the relation "is greater than" between two numbers as "to the right of" on the number line.)

Or, we could keep the normal ordering, but throw away a distance property as on the number line below.

2.



(Here the child would not be able to "take a trip" from $^{+}2$ to $^{+}5$ and have it be the same distance as "a trip" from 0 to $^{+}3$, or as "a trip" from $^{-}2$ to $^{+}1$.)

In marking points, it is very unlikely that children will throw away the order as in number line 1; however, young children, if given complete freedom the first time, will occasionally come up with a number line like example 2. In the second film excerpt, the teacher may have avoided this possibility by marking the location of seven or eight points himself (properly spaced to preserve the distance property) before he asks the children to mark any points.

Agenda: Item 9. Discussion of Third Film Excerpt

1. There are a very large number of uses of graphs in the school program, beginning perhaps even as early as kindergarten. One kindergarten teacher has children plant seeds, and each day they tear a strip of paper at the height of the plant as it grows. If these strips of paper are then pasted side by side, as shown in Diagram 1, they provide a kind of "graph." This kind of "graph," which is made from actual physical objects (such as the strips of paper here) is an example of what is nowadays often called "self-graphing material."

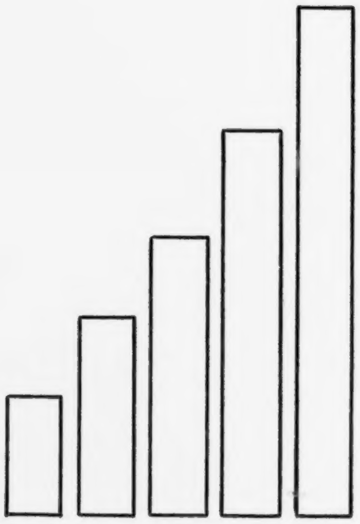


Diagram 1

At a later grade level, this same experiment with growing plants might be performed, but with numerical records where the days are recorded along the horizontal axis, and the children measure the height of the plants using rulers, and record this height (perhaps in centimeters, or else in inches) as the vertical co-ordinate. This is illustrated in Diagram 2.

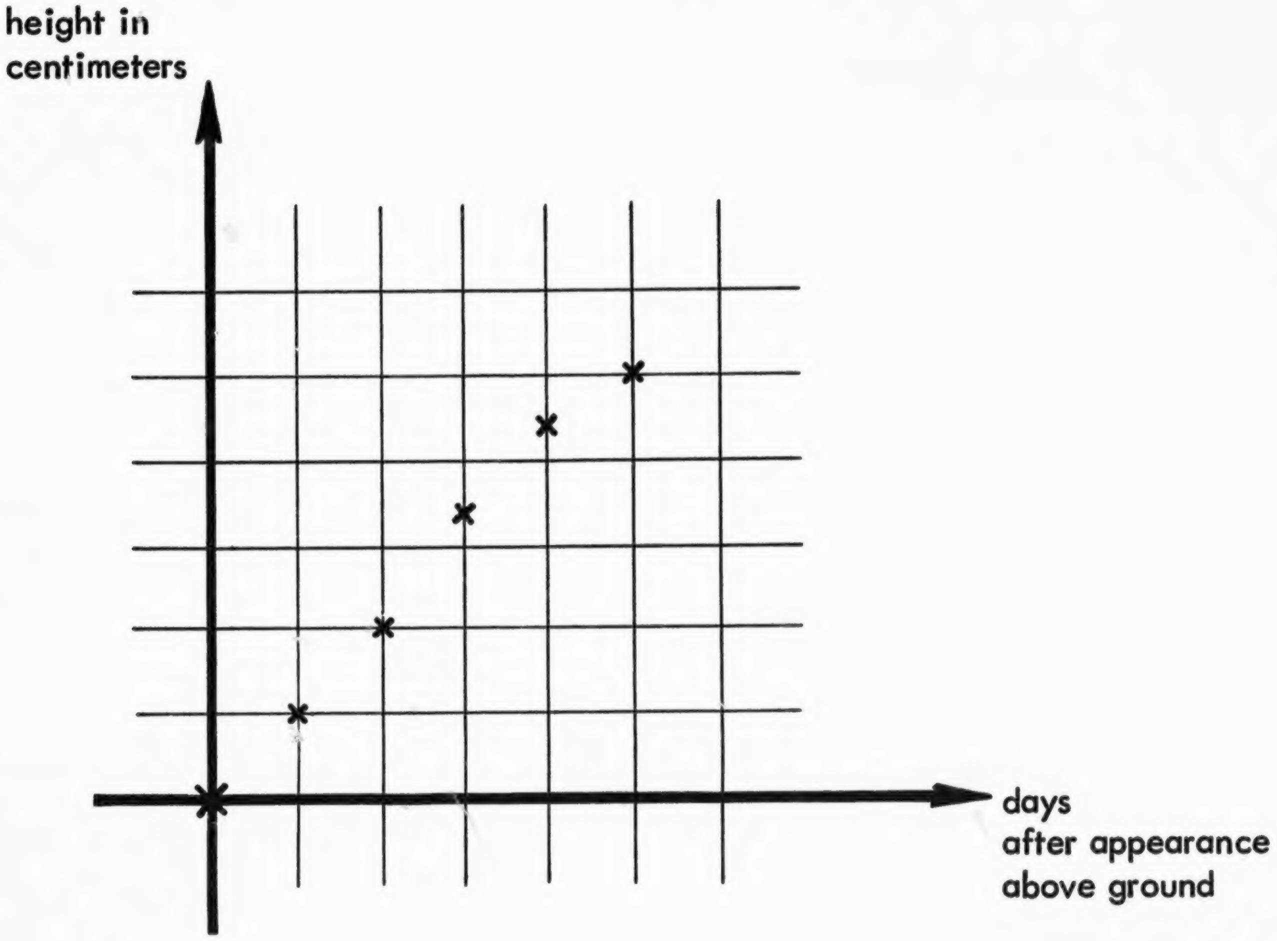


Diagram 2

Once such a diagram has been started, one can ask such questions as "In how many days will the plant be seven centimeters tall?" and so on. This kind of question doesn't have any "exact" answer, of course, since it will depend upon temperature and sunlight in the ensuing time (among other things), but it shows how reasonable estimates can be made by using a graph.

Graphs of economic phenomona of course appear every day on the front page of the Wall Street Journal, and in many other newspapers.

Graphs can also be used in connection with social studies work, for example to show the growth of population within your town or city as a function of the date, to show average income for your city or town as a function of the year, and so on.

One can even use graphs in recording children's progress in class work in various areas.

Many other possibilities will doubtless occur to you.

2. We seem to have answered this under Item 1 above.

General Remarks and Background Information

Please read this section before coming to the Second Session.

The present in-service course for teachers is intended to help broaden the mathematical content in grades 2 - 8. Specifically, it is intended to help teachers at these grade levels to teach a supplementary program of important mathematical ideas: the arithmetic of signed numbers; variables; functions; and Cartesian co-ordinates.

The program contained herein is supplementary; it does not replace a standard arithmetic program, although use of the supplementary materials will usually modify, improve, and strengthen the "basic" arithmetic program.

These supplementary topics may be kept separate from existing school programs, or they may be assimilated within either the arithmetic or the science programs. In the next few pages, we shall discuss these topics on the assumption that they are being kept separate and distinct from the basic science and arithmetic programs.

I. Informal Exploratory Experiences.

The mathematical experiences for children that have been devised by the Madison Project are novel. Indeed, they are so novel that it has seemed desirable to give them a distinctive name, for they do not look like most people's idea of "usual" lessons. Consequently, we have called the experiences described here by the name "informal exploratory experiences."

The Madison Project has, for a number of years, been engaged in developing these mathematical activities for children. Many of them have been recorded on films. These films show actual classroom lessons, without subsequent film editing, exactly as the lessons occurred. Short excerpts from these films are included in the present in-service course. If you wish to gain a better idea of the Madison Project's "typical" lesson, you may wish to view some of the films in their full-length form.

1. Two Kinds of Lessons. The "informal exploratory experiences" are generally one of two types: either they take the form of "using, exploring, and becoming familiar" with some ideas, or else they take the form of informal seminar-type discussions by teacher and students. Some lessons partake of both natures, and might be hard to classify.

The lessons involving "using, exploring, and becoming familiar" we call experience lessons. As an example, the film Experience in Estimating and Measuring Angles[1] shows a lesson where the teacher draws a picture of an angle on the blackboard, the children guess its measure in degrees, and then they measure it, with a protractor, to see whose guess was closest.

The point of this lesson is probably clear: there is no specific "discovery" that we want the children to make. One might say there is no sharply-defined point to this lesson at all. (Certainly, we would never grade the children according to how close they could guess the measure of an angle!) We just want the children to feel a comfortable familiarity in their future dealings with angles.

By contrast, the film Second Lesson shows (among other things) the introduction of the idea of "linear graphs," which we shall study later in this in-service course. Here there is discussion by the teacher and students (as well as marking points on the graph, which is done by the students). There is, as we shall see presently, a well-defined "discovery" that we want the children to make (indeed, there are two or three worthwhile "discoveries"). The teacher tries to guide the discussion unobtrusively so that as many of the children as possible make the "discovery" for themselves. In any event, the teacher does not tell the students what the "discovery" is.

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[1] The film Experience in Estimating and Measuring Angles is available (as are the other films mentioned in these notes) from The Madison Project, Webster College, St. Louis, Missouri 63119.

Since an important aspect of "informal exploratory experiences" is the attempt to develop creative student behavior, we shall turn to this matter in the next section.

2. Creative Student Behavior. We have developed certain methods for handling these "informal exploratory experiences" which appear to increase student motivation, and to cause students to approach mathematics more creatively. What is the best way to "teach" these lessons? We would suggest the following:

i) Wherever possible, withhold value judgements. Students become conservative if they feel they are being judged; they become more creative when they feel they are being appreciated. We advocate looking impressed by honest student achievements -- which actually occur rather often if we watch for them carefully-- but we advocate avoiding (as much as possible) the use of words like "wrong," "incorrect," and so forth.

ii) Employ the "light touch" -- we often introduce an idea lightly, and drop the matter, at least for the present, before the students are bored with the idea. The teacher can return to the topic in later lessons. We believe several "light" exposures of an idea are more effective, in the long run, than one "thorough" one. Perhaps the trouble with "thorough" presentations is that, coming one after another, they are boring and *not* very stimulating.

iii) We seek to get *student participation* wherever this can be achieved without destroying the continuity of the lesson. This can be pursued in many situations: for example, if you are doing addition problems, *let the children make up the problems*. They are then more highly motivated to solve them.

iv) The "informal exploratory experiences" are in fact *informal*. Just how you arrange this we shall leave up to you, but it is our firm conviction that a generally formal tone discourages student creativity, whereas an informal tone to the class encourages student creativity.

v) Perhaps this recommendation is a corollary of the previous remarks: we recommend that *the teacher attempt to play a generally unobtrusive role* in the lessons. Again, what this can mean we leave up to you; some ideas can be gleaned from viewing the Madison Project films.

vi) Since we wish to emphasize the role of student discovery, many experienced Madison Project teachers recommend the device of *naming new discoveries after the children who make them*, as in "Mary's method for solving quadratic equations," "Dan's equation," "Jerry's method," and so on.

vii) One important matter is the *degree of teacher control over class discussion*. If this is too great -- the most common teacher error -- then discussion does not flourish; on the other hand, if the teacher provides too little guidance, the discussion can flounder.

The difference is often determined by the precise form in which the teacher asks questions. For example, in a lesson which we just studied, we are concerned with putting pebbles into a bag. Some teachers, holding the bag, ask "what shall I do now?" The result can be disorganized chaos. If, instead, the teacher asks: "How many pebbles shall I put into the bag?" the student response is far more satisfactory. The children have not lost the feeling of freedom of choice, but they have received enough guidance to keep the discussion headed in a fruitful direction.

II.. What is this "New Mathematics"?

Both in our society generally, and in our schools and colleges, one hears a great deal nowadays about "new mathematics." What is this talk all about?

It is useful to look at a few aspects of what is called "new" in mathematics:

i) Routine processes are disappearing as human tasks. Over the past century or more, we have seen routine physical tasks eliminated as jobs for people, and re-assigned as tasks for machines. (Nowadays we even have electric can-openers, which probably follow in the tradition of reapers, binders, electric starters for automobile engines, steam shovels, self-stoking furnaces, and electric ice cream freezers.) More recently--and especially since World War II -- we have witnessed the re-assignment of routine mental tasks: these, too, are now the proper task of machines. They are rapidly disappearing as a proper task for human beings.

Indeed, computing machine experts say that, if you can describe a task precisely, a machine can perform it.

So extreme is this shift that it already effects the gross employment situation in the United States. For the first time in our history we have unemployment at the same time that we have unfilled jobs: but the unemployed people are at the routine end of human performance, and the unfilled jobs are at the sophisticated and creative end.

Consequently, educating people in routine skills will surely diminish in importance in future years, and educating people in creative and sophisticated methods of thought will become increasingly important.

ii) In particular, arithmetic is rapidly being automated--that is, reassigned from people to machines. The digital computer, the slide rule, and the desk calculator all have a hand in this; so do the cash registers which automatically make the correct change, without requiring the cashier to do any arithmetic. (The rapidly increasing use of credit cards may also contribute to a growing automation in this area.) Indeed, some cash registers not only automatically make the correct change, they automatically dispense the correct number of green stamps.

iii) Many important branches of mathematics that have been developed in recent years have not yet appeared in the school curriculum, and need to do so. Among these are: logic, which is the systematic procedure for starting with a collection of statements and extracting from them other statements. This process is important in law, in philosophy, in mathematics and elsewhere. Also, statistics, which is a systematic procedure for starting with uncertain, incorrect, or incomplete data and obtaining as much further information as possible.

Both logic and statistics are examples of relatively new branches of mathematics (both having been developed mainly in the twentieth centry) that are just now beginning to find their way into the school mathematics program.

iv) The process described above is probably part of a general broadening of the school curriculum. Our "traditional" curriculum--not merely in mathematics, but in any area--included a very narrow slice of the important knowledge in that area. What is worse, these narrow slices were somewhat arbitrarily selected, and are often not good choices for twentieth-century children. In English we have usually omitted rhetoric and linguistics. In social studies we have focussed on Europe and North America, and completely eliminated South America, Africa, and Asia, and we have omitted the history of the major religions. In modern languages we have focussed on French, German, and Spanish; we have almost completely ignored Russian, Chinese, Japanese, Swahili, and all other contemporary languages. In music we have focussed on the "harmonic common-practice period" of "classical" music--from Bach to Brahms--and have omitted pre-Bach music, modern music, jazz, folk music, oriental music, ethnic music, and contemporary "background" music. In every one of these instances the exclusion is complete and absolute; we act as if the excluded areas do not exist, or are of absolutely no importance. Consequently, when the United States encounters matters that are important in Korea, or Formosa, or Vietnam, or the Congo, our citizens are not prepared to think about such things. We have failed to build the broad maturity which is necessary in a democracy.

A general broadening of the curriculum is clearly in order. For mathematics, this means going far beyond the narrow confines of adding, subtracting, multiplying, dividing, and retail purchasing.

v) Finally, the rapidly expanding amount of knowledge in the world today is itself a factor demanding new educational orientations. One major corporation reports that it is often cheaper and faster to repeat a scientific study than it is to hunt through their files to see if the study has already been carried out. This tendency can only grow. For education, this means that "remembering" will have to yield ground to the processes of "learning" and "discovering." The task of today's child is to learn how to learn.

Teachers studying the present material should notice that some of the "newness" comes from modern efforts at rationalizing notation, definitions, and so on. A few years ago we learned these in the forms in which they developed historically: most of them, one might say, "just grew." For example, the single symbol "- " was used for three different ideas:

subtraction: $5 - 2$

negative numbers: -2

additive inverses: $x + (-x) = 0$

This could only lead to confusion, and did. Today these three ideas have separate names and separate symbols, which we shall study in the following materials:

subtraction: $5 - 2$

negative numbers: ${}^-2 \times {}^-3 = {}^+6$

additive inverses
(or "opposites"): $\square + {}^{\circ}\square = 0$

$${}^{\circ}({}^+2) = {}^-2$$

$${}^{\circ}({}^-3) = {}^+3$$

Consequently, we suggest that you be patient with those things which "look different" from the way we ourselves learned them in school. See if you can recognize the reasons for the changes in definitions, notations, sequential order of topics, and so on. In nearly all cases we believe you will find the new notations simpler and more satisfactory once you understand them.

SUPPLEMENTARY MODERN MATHEMATICS

For Grades 1 - 9

An In-Service Course for Teachers

SECOND SESSION

Note: A brief review of last week's work is included at the end of this session. Its use is optional with you.

Agenda:

1. True, False, and Open
2. First Film Excerpt: True, False, and Open
3. Open sentences with Two Variables
4. Second Film Excerpt: Open sentences with two variables
5. Graphs for Truth Sets
6. Third Film Excerpt: Introduction to Linear Graphs
7. Discussion of Film Excerpts
8. Plotting Points in Four Quadrants
9. Fourth Film Excerpt: Tic Tac Toe in Four Quadrants
10. Optional -- Discussion: What Can I take Back to My Class This Week?
11. Review -- Optional: Pebbles-in-the-Bag (included as a final fifth film excerpt)
12. Optional: Play Tic Tac Toe Yourselves!

1. True, False, and Open

This is a simple mathematical idea, but an important one. It is a "modern" approach, and differs from what was traditionally done.

i) "True statements. Any of the following are "true" statements:

written	read
$2 + 2 = 4$	two plus two equals 4
$37 + 13 = 50$	thirty seven plus 13 equals 50
$2\frac{1}{2} + 2\frac{1}{2} = 5$	$2\frac{1}{2}$ plus (or "and") $2\frac{1}{2}$ equals 5
$7 < 12$	7 is less than 12
$5 = 5$	5 equals 5
$3 \neq 7$	3 is not equal to 7

There is probably nothing new here, except perhaps the symbols < ("is less than") and ≠ ("is not equal to"). The uses of these 2 symbols are probably reasonably clear.

ii) "False" statements. "Modern" mathematics teaching makes much more use of false statements. Notice that they can be useful and reliable, provided we know that they are false. The following statements are all false:

written	read
$2 + 2 = 5$	two plus two equals 5
$4 \neq 4$	four is not equal to four
$7 < 5$	seven is less than five
$3\frac{1}{2} + 1\frac{1}{2} = 4\frac{1}{2}$	$3\frac{1}{2}$ plus $1\frac{1}{2}$ equals $4\frac{1}{2}$

Again, there should be nothing here that is really new.

iii) "Open sentences. This name is new, although the idea is not. These are called "open sentences":

________ is the teacher of this class.

$$3 + \square = 5$$

$$3 < 1 + \square < 8$$

$$\square \times \square = 4$$

iv) An open sentence contains a "pronumeral" or a "variable" (mathematicians ordinarily use the second of these names), which may be written ________ or □ or △ or x or A or N, etc. When we "substitute numbers into the □," we get a resulting statement that may be false (e.g. $3 + \boxed{4} = 5$) or which may be true (as in $3 + \boxed{2} = 5$).

v) In substituting into variables, we agree to observe the "rule for substituting," which says that, whatever number we substitute into the 1st occurrence of □, we must substitute this same number into all other occurrences of the □. (An analogous rule applies to △, A, x, etc.)

Thus

$$\boxed{2} \times \boxed{2} = 4$$

is a correct use of the "rule for substituting," but

$$\boxed{4} \times \boxed{1} = 4$$

is an incorrect use of the "rule for substituting."

Notice that, since we have two independent criteria, there are four possible cases:

	rule for substituting	truth value of statement
$\boxed{2} \times \boxed{2} = 4$	used correctly	true
$\boxed{5} \times \boxed{5} = 4$	used correctly	false
$\boxed{4} \times \boxed{1} = 4$	violated	true
$\boxed{5} \times \boxed{3} = 4$	violated	false

vi) For any open sentence, the set of all numbers which, when substituted legally for the variable, produce a true statement, is called the truth set of the open sentence. This truth set is customarily written using "braces" or "wiggly brackets," e.g.

open sentence	truth set
$3 + \square = 5$	$\{2\}$

YOU MAY NOW WISH TO VIEW THE FIRST FILM EXCERPT.

3. Open Sentences with Two Variables

Besides open sentences with a single variable, such as

$$3 + \square = 5$$

and

$$\square + \square = 10,$$

it is also possible to write open sentences with two or more variables, such as

$$\square + \triangle = 10.$$

In this case, it is extremely important to notice carefully what the "law for substituting" *does* say, and what it *does not* say. To put it briefly, the "rule for substituting" says:

i) If the shapes are the same, the numbers that you substitute must be the same.

ii) If the shapes are different, the *numbers may be either the same*, *or else different*. It is easier to see this in examples:

	use of "rule for substituting"	truth value of resulting statement
$\boxed{6} + \triangle\!\!\!\!\!{\scriptstyle 4} = 10$	correct	true
$\boxed{9} + \triangle\!\!\!\!\!{\scriptstyle 3} = 10$	correct	false
$\boxed{4} + \triangle\!\!\!\!\!{\scriptstyle 6} = 10$	correct	true
$\boxed{5} + \triangle\!\!\!\!\!{\scriptstyle 5} = 10$	correct	true
$\boxed{6} + \triangle\!\!\!\!\!{\scriptstyle 6} = 10$	correct	false

When an open sentence contains two different placeholders (or "variables"), we must specify *two* numbers for each substitution. By a traditional convention, the number we say *first* is to be substituted into the $\square$, and the number we say *second* is to be substituted into the $\triangle$. Thus, for the open sentence

$$\square + 5 = \triangle ,$$

the substitution (4,9) would mean

$$\boxed{4} + 5 = \triangle\!\!\!\!\!{\scriptstyle 9}\;$$

whereas (9,4) would mean

$$\boxed{9} + 5 = \triangle\!\!\!\!\!{\scriptstyle 4}\;$$

Evidently, then, the *truth set* for such an open sentence would consist of *pairs* of numbers, where *the order is important*.

Example 1. For

$$\square + 5 = \triangle$$

the truth set includes these ordered pairs

(0,5)

(1,6)

(2,7)

(3,8)

(4,9)

(5,10)

(6,11)

and so on. Note that (9,4) is *not* in the truth set.

It is often convenient to list the truth set by a *table*, like this:

□	△
0	5
1	6
2	7
3	8
4	9
5	10
6	11
7	12
⋮	⋮

(as usual, the concluding three dots mean that the list goes on without stopping).

Example 2. For the open sentence

$$(1 \times \square) + 3 = \triangle,$$

the truth set includes these pairs of numbers

□	△
0	3
1	4
2	5
3	6
4	7
5	8
⋮	⋮

YOU MAY NOW WISH TO VIEW THE 2ND FILM EXCERPT.

5. Graphs for Truth Sets

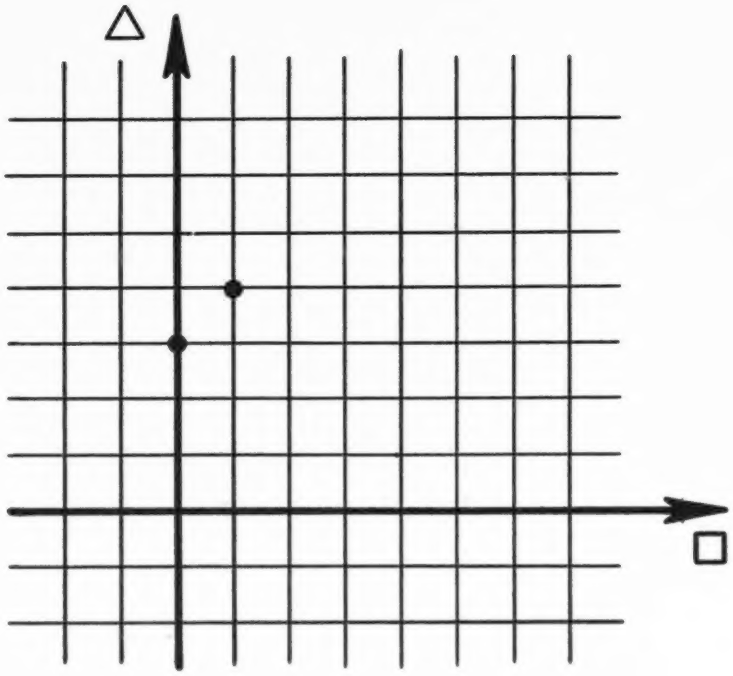
1. For the open sentence

$$(1 \times \square) + 3 = \triangle,$$

the truth set includes these ordered pairs of numbers:

□	△
0	3
1	4
2	5
3	6
4	7
.	.
.	.
.	.

The first two of these pairs (0,3) and (1,4) have been marked on this graph. Can you mark the next three pairs on the graph?



(Note that we start our counting at the heavy lines, which are called "axes." Counting upward or to the right represents positive numbers; counting down or to the left represents negative numbers.)

2. Have you discovered any "secret pattern" for problem 1?

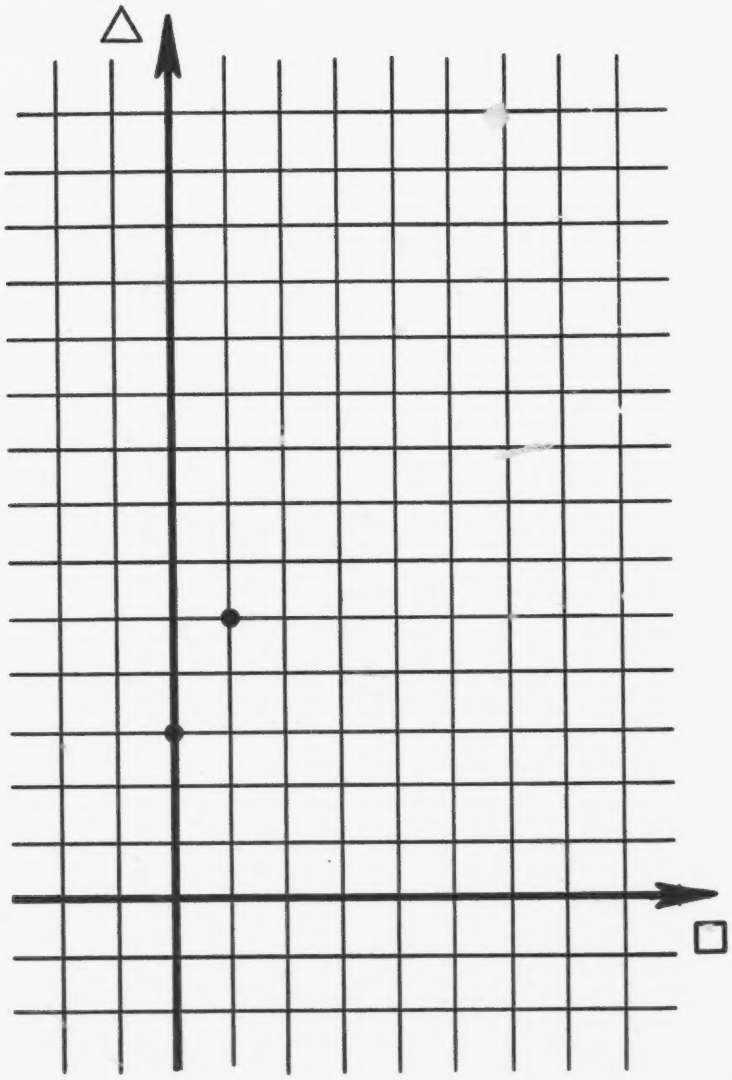
3. For the open sentence

$$(2 \times \square) + 3 = \triangle,$$

the truth set includes these ordered pairs of numbers:

$\square$	$\triangle$
0	3
1	5
2	7
3	9
4	11
5	13
.	.
.	.
.	.

The first two of these pairs have been marked on the following graph. Can you mark the other pairs on the graph?



4. Have you discovered any "secret pattern" for problem 3?

YOU MAY NOW WISH TO VIEW THE THIRD FILM EXCERPT.

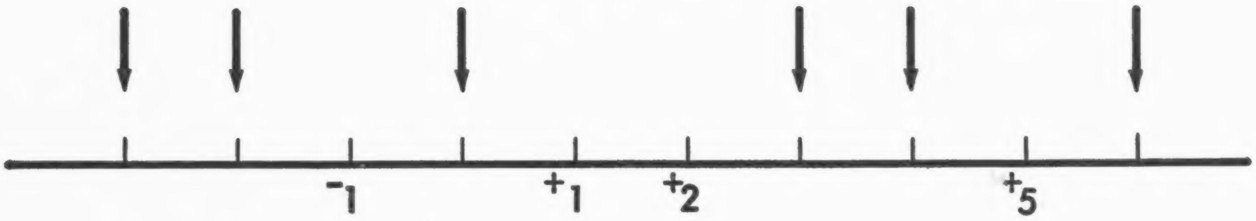
7. Discussion of Film Excerpts

We leave this up to you. You may wish to refer to the book:

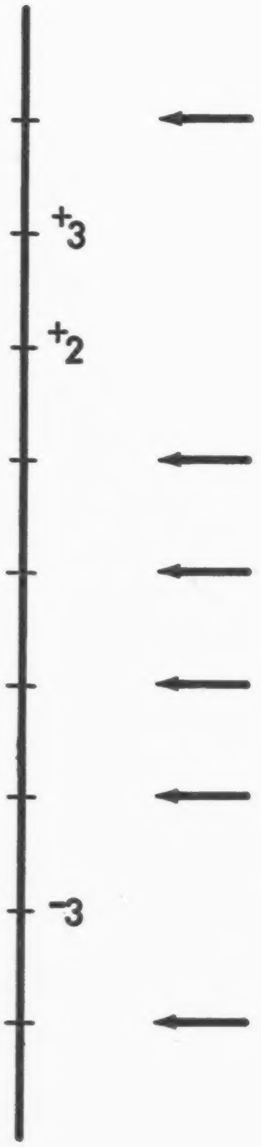
Robert B. Davis, Discovery in Mathematics, Addison-Wesley, Inc., Reading, Massachusetts, 1964.

8. Plotting Points in Four Quadrants

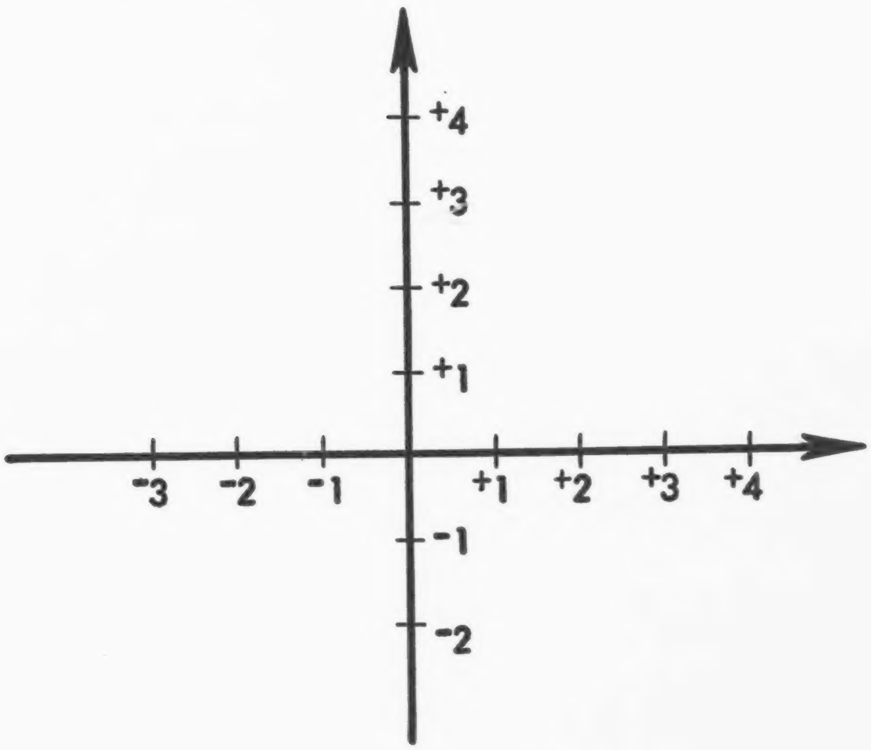
1. On this horizontal number line, can you fill in the number labels for the points indicated by arrows?



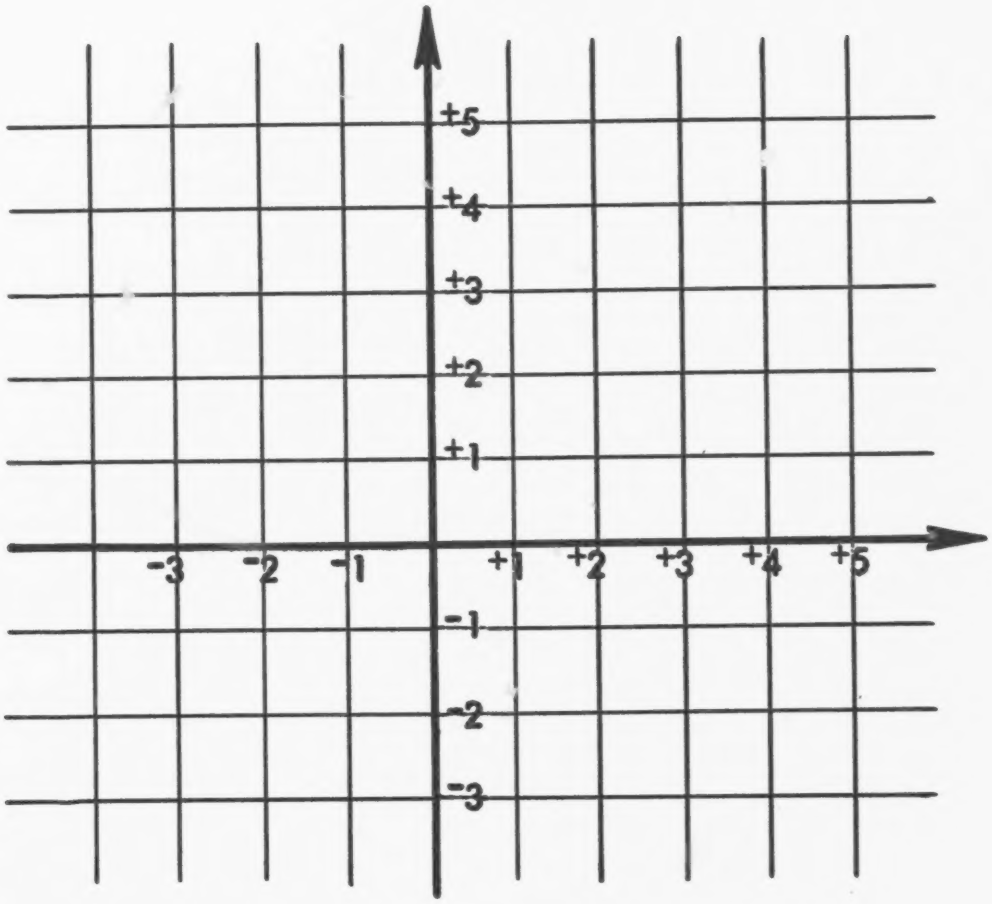
2. On this vertical number line, can you fill in the number labels for the points indicated by arrows?



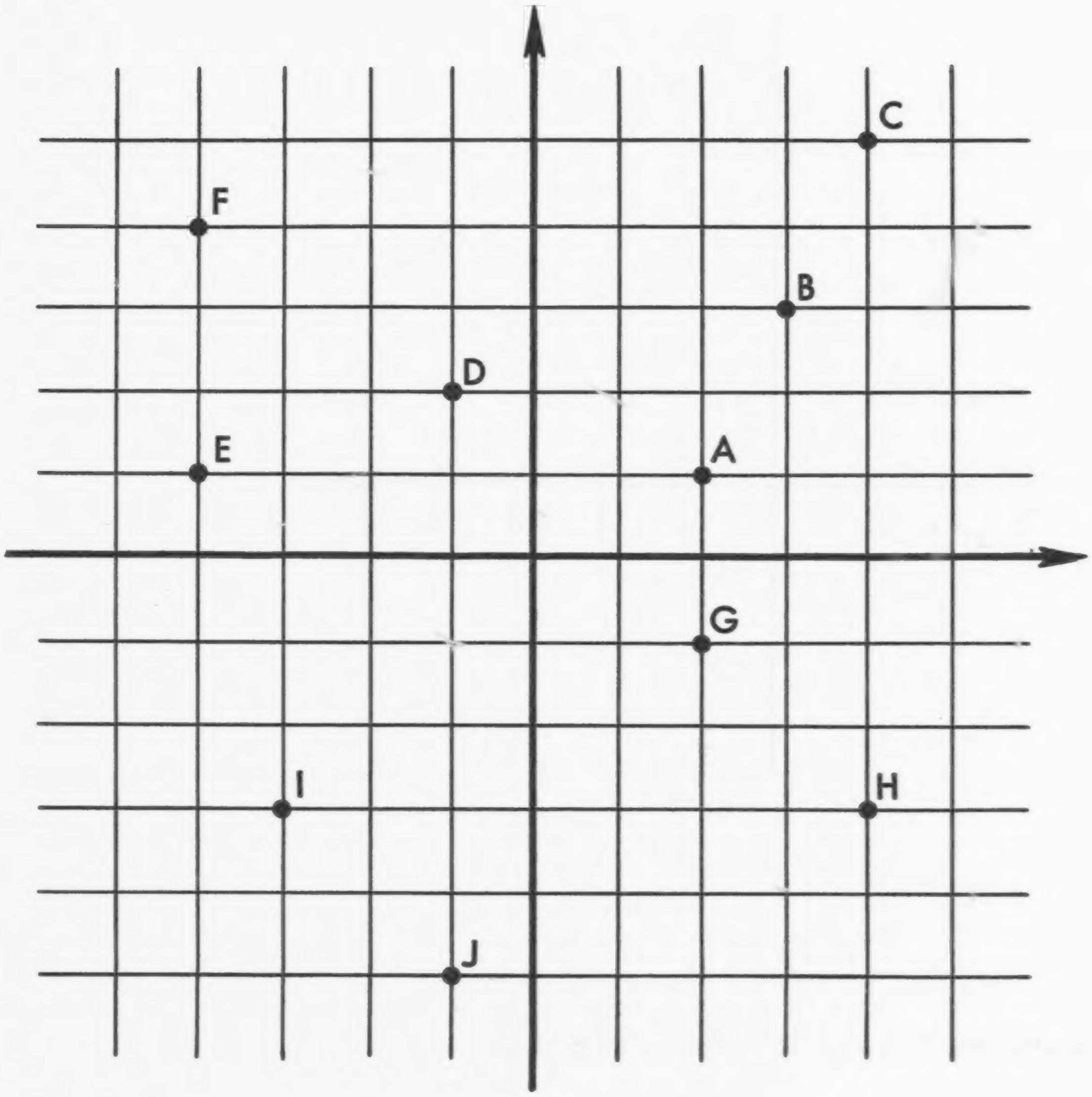
3. We can cross two number lines like this:


+4
+3
+2
+1
-3
-2
-1
+1
+2
+3
+4
-1
-2

or, to make it easier for the eye to follow, like this:


+5
+4
+3
+2
+1
-3
-2
-1
+1
+2
+3
+4
+5
-1
-2
-3

Can you write the coordinates of the points A, B, C, D, E, F, G, H, I, and J?



The answers are given in the yellow reference booklet for the second session.

4. Can you mark these points on the graph?

($^+1$,$^-1$)

($^+2$,$^+5$)

($^-1$,$^-1$)

($^+3$,0)

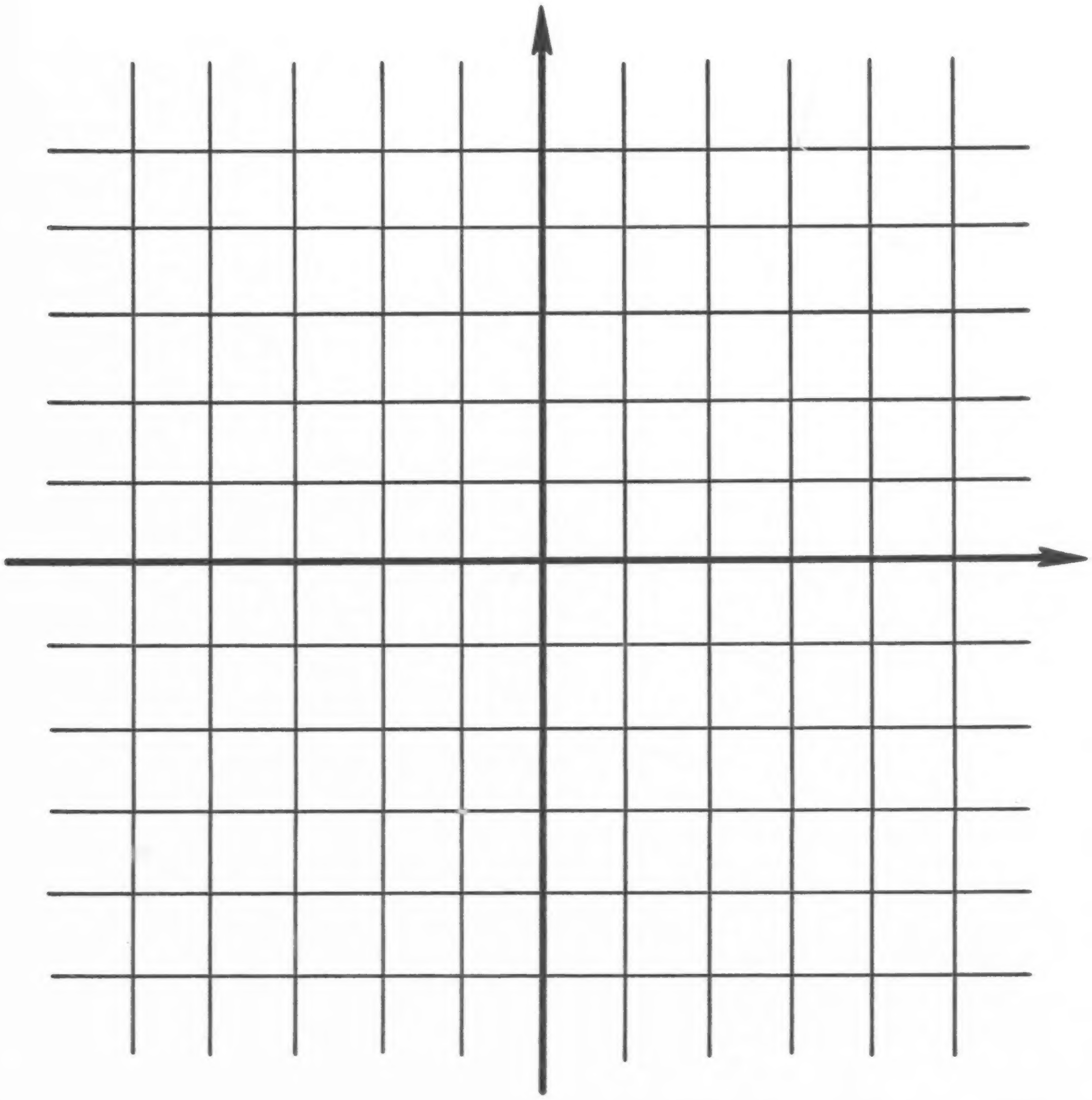
(0,$^-2$)

($^+4$,$^-2$)

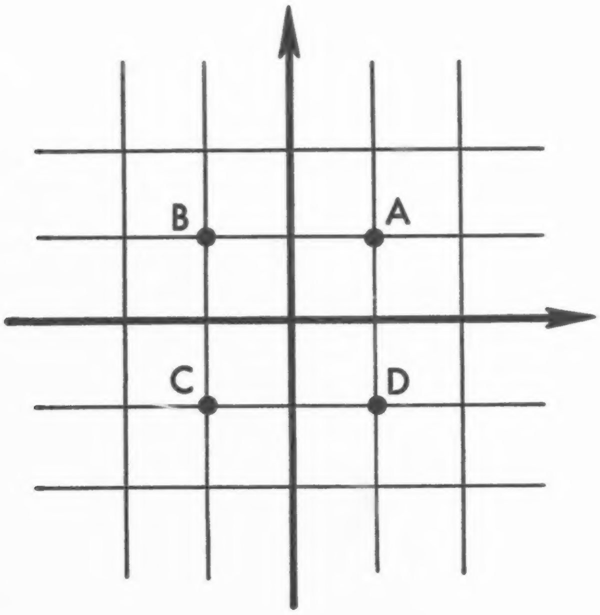
($^-4$,$^-1$)

($^-3$,$^+2$)

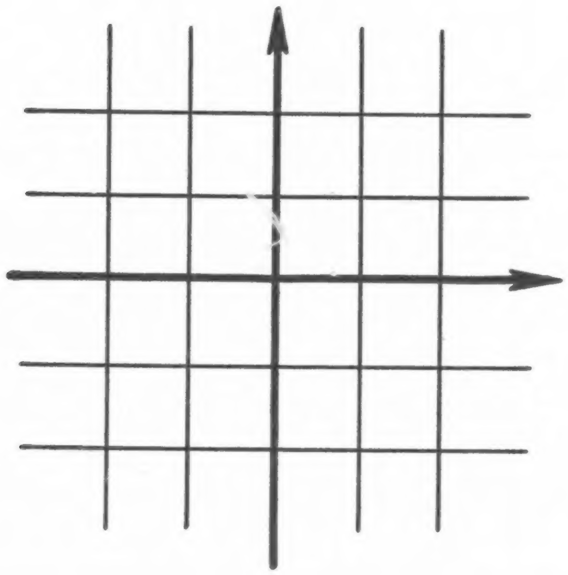
Remember: The number given first always refers to the horizontal number line, and the number written second always refers to the vertical number line.



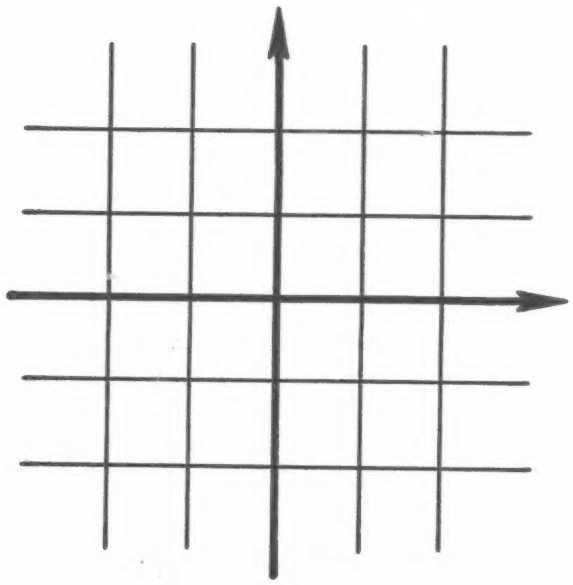
5. Can you find the co-ordinates of the four points A, B, C, and D?



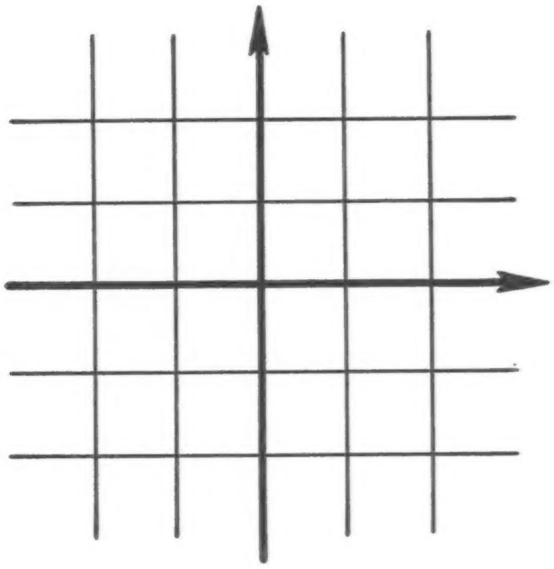
6. Can you mark the point ($^-1$, $^-1$) on a graph?



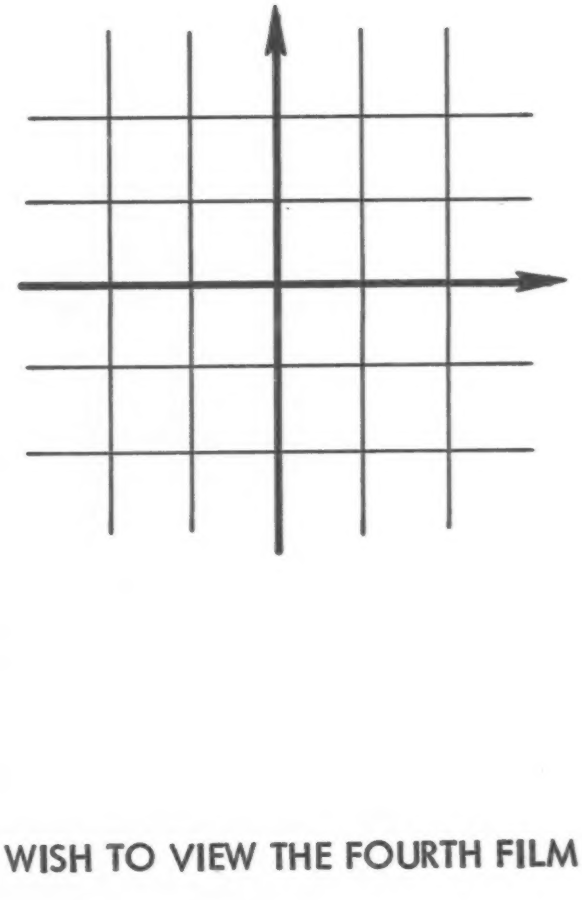
7. Can you mark ($^+1$, $^+1$) on a graph?



8. Can you mark ($^+1$, $^-1$) on a graph?



9. Can you mark ($^-1$, $^+1$) on a graph?



YOU MAY NOW WISH TO VIEW THE FOURTH FILM EXCERPT.

10. What Can I Take Back to My Class This Week?

We leave this discussion up to you.

11. Pebbles-in-the-Bag: Review

This has been included as an optional 5th Film Excerpt. You may also want to have one of the group teach it as she would to children.

12. You may want to play Tic Tac Toe yourselves. Use either one quadrant, or all four. We suggest that you use a 5-by-5 "board," and require 4 marks (by the same team) in an uninterrupted row to win.

MADISON PROJECT
Syracuse University · Webster College

SUPPLEMENTARY MODERN MATHEMATICS

For Grades 1-9

REFERENCE BOOK

Second Session

Agenda Item 1. True, False, and Open

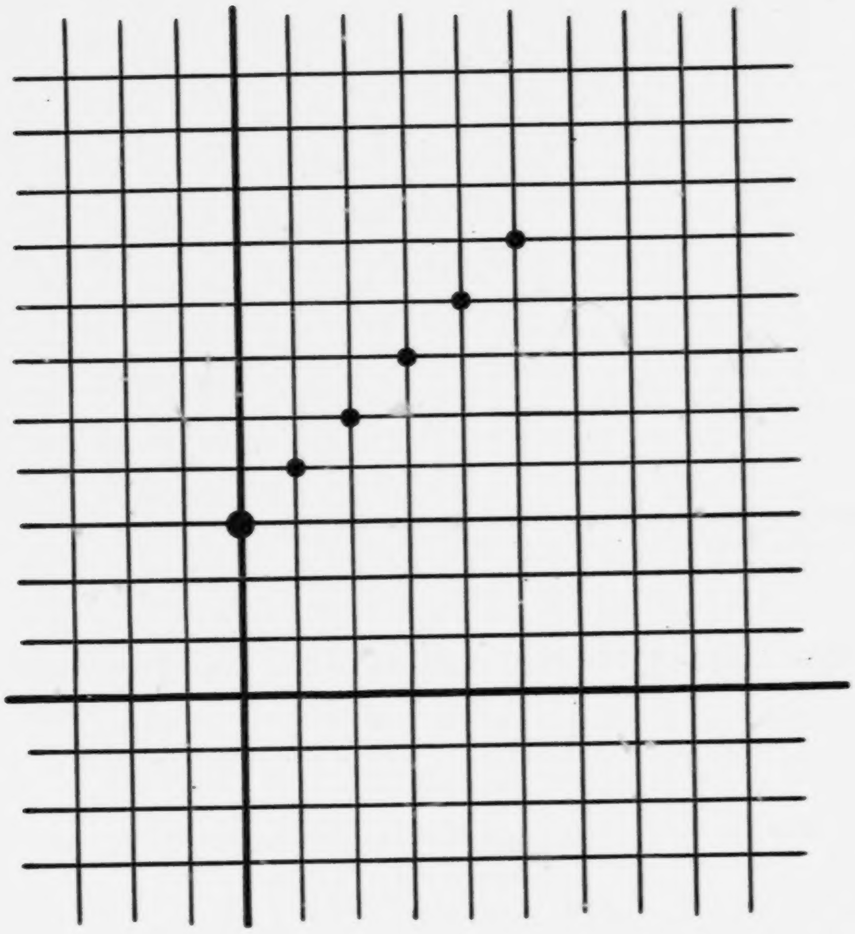
The "Rule for Substituting"

The "rule for substituting" really represents a tacit agreement. This agreement is understood to be as follows: If the writer of an open sentence displays the same variable, e.g. □, more than once in a sentence, he indicates that he was considering only those statements formed by substituting the same element of the replacement set in each occurrence of □; i.e., he means that if you put 4 in one box, you are to put 4 in every box in the open sentence. A "legal" substitution is one in which you follow the writer's intent. If the writer intends that there be two variables in a sentence, and he wishes to consider statements formed by unrestricted substitution in each variable, he uses 2 symbols; e.g., □, △.

We can "break" the "rule for substituting," but when we do we are not considering the same open sentence as the writer did. It goes without saying that we must observe the same tacit agreement when writing open sentences, if we are to be interpreted correctly.

Agenda Item 5. Graphs for Truth Sets

1. $(1 \times \square) + 3 = \triangle$



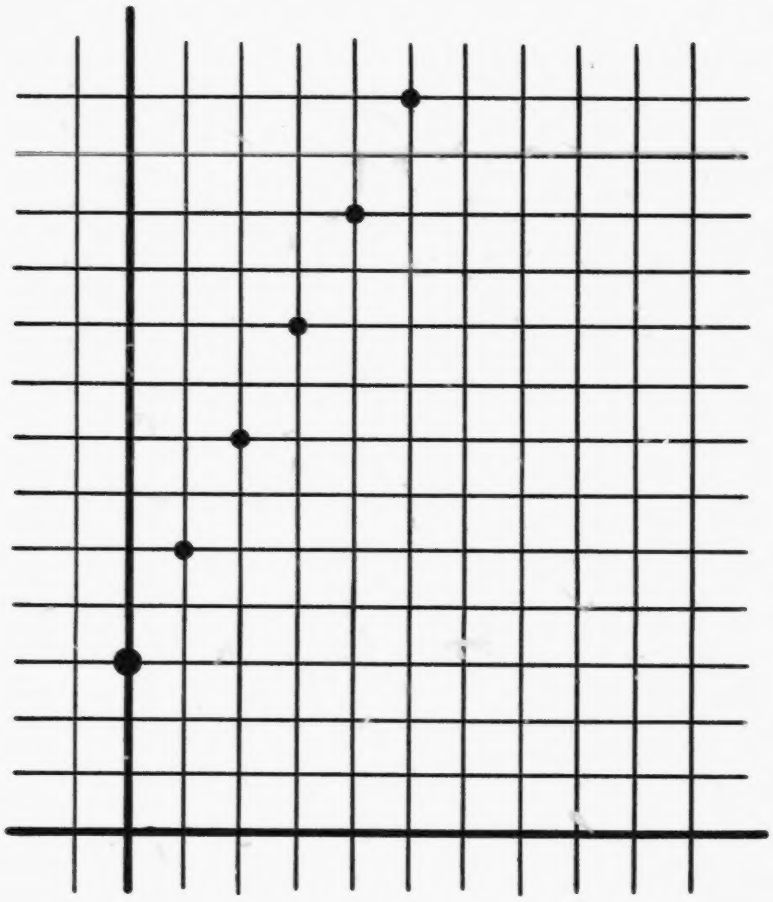
You will probably have noticed that the additional points lie in a straight line when plotted.

If you have found more ordered pairs in the truth set, you may enjoy plotting them too. You should now be able to find points which are in the truth set of the equation by inspection of your graph alone.

See if you can, and then check the ordered pairs that you think will work by substitution in the equation. There is a pattern.

2

2. (2 x ☐) + 3 = △



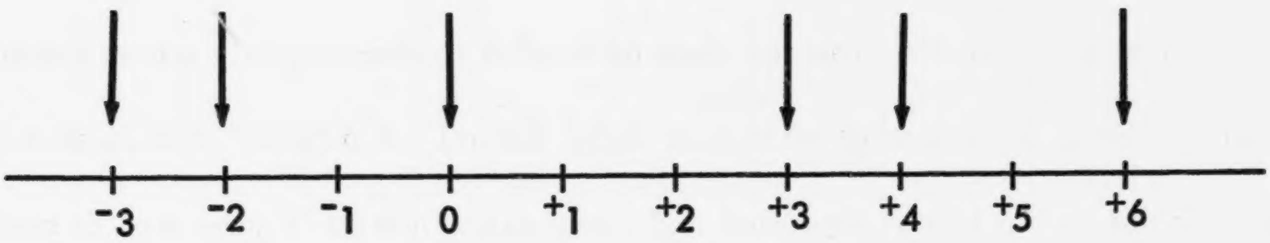
Again, you will have noticed that the location of the points in the truth set of (2 x ☐) + 3 = △ is not haphazard. They do have a pattern and lie in a line characteristic of this type of equation. They are very appropriately named linear equations. You or your students may enjoy finding additional points in the truth set of (2 x ☐) + 3 = △ by inspection of the graph. Test your selections in the equation.

Another idea which presents itself would be to plot the truth set of several linear equations on the same page. You could start with (1 x ☐) + 3 = △ which you already have, then put (2 x ☐) + 3 = △ on next, using another color pencil or different symbol to mark the ordered pairs. Put (3 x ☐) + 3 = △ on next and then perhaps (4 x ☐) + 3 = △.

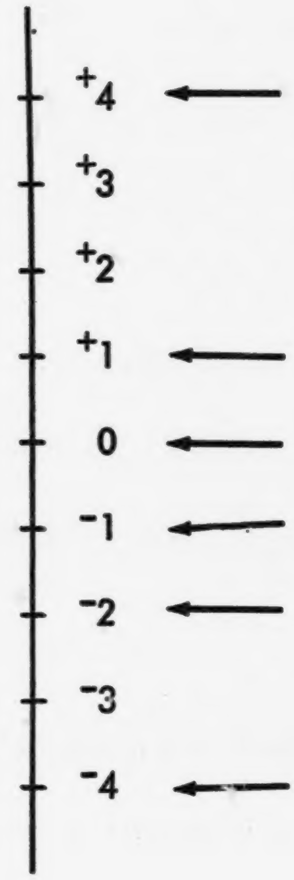
Several interesting items appear from this activity and students seems to grasp the situation quickly. Among other things they will probably notice that the "steepness" or "slope" of the line is affected by the number which multiplies the ☐ . Do you think this type of work could best be done at the board or with groups of students at tables?

Agenda Item 8. Plotting Points in Four Quadrants

1. Horizontal number line



2. Vertical number line



Two or three integers initially placed on the line seem adequate for most classes to begin with. Careful spacing of "hash" marks on a neat line encourages careful thinking and will aid later when students are asked, for example, to name the number indicated by an arrow midway between $^+2$ and $^+3$. The important idea here is to give students the chance to discover a way to order the integers on the line. We could ruin their discovery entirely by giving them the completed picture. . . .

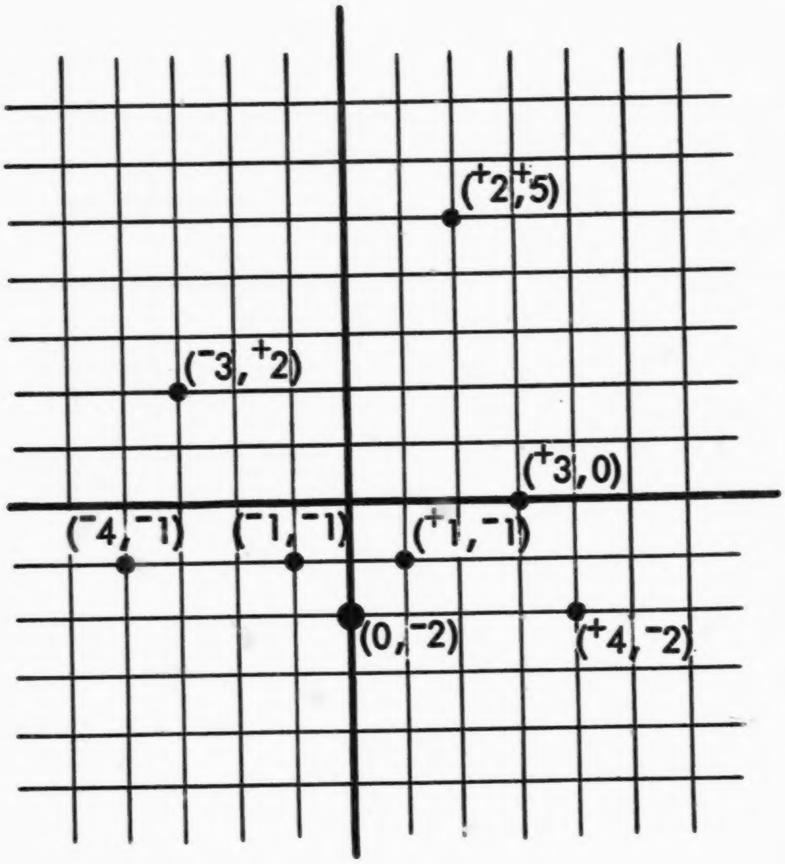
As the teacher questions the class for ideas as to what number properly names a particular point on the line, disagreement often runs high. Let it! If a "poor" choice of number to name a point on the line is suggested and used, extra interest is generated as another student catches the "error." "Poor" choices and "errors" can contribute much to discovery.

3. The coordinates for the points A - J are as follows.

A. $(^+2, ^+1)$
B. $(^+3, ^+3)$
C. $(^+4, ^+5)$
D. $(^-1, ^+2)$
E. $(^-4, ^+1)$
F. $(^-4, ^+4)$
G. $(^+2, ^-1)$
H. $(^+4, ^-3)$
I. $(^-3, ^-3)$
J. $(^-1, ^-5)$

If your answers don't check with those above, you may want to review page 15 of this second lesson. Remember: order is important. In ordered pairs, the first number refers to position to the right or left of the vertical axis. The second number refers to position up or down from the horizontal axis.

4. The suggested points are marked on a graph below. This clever way of relating number to position on the plane is very important in much of higher mathematics. Calculus, geometry, and algebra itself owe a debt to Descartes and his Cartesian Plane. The pair of numbers associated with a point on the plane are called its Cartesian coordinates, or more simply, its coordinates. With every pair of numbers is associated a point; with every point a pair of numbers. This is truly a powerful tool!



(⁺2,⁺5)
(⁻3,⁺2)
(⁺3,0)
(⁻4,⁻1)
(⁻1,⁻1)
(⁺1,⁻1)
(0,⁻2)
(⁺4,⁻2)

5. The coordinates of the four points are as follows:

A. (⁺1,⁺1)

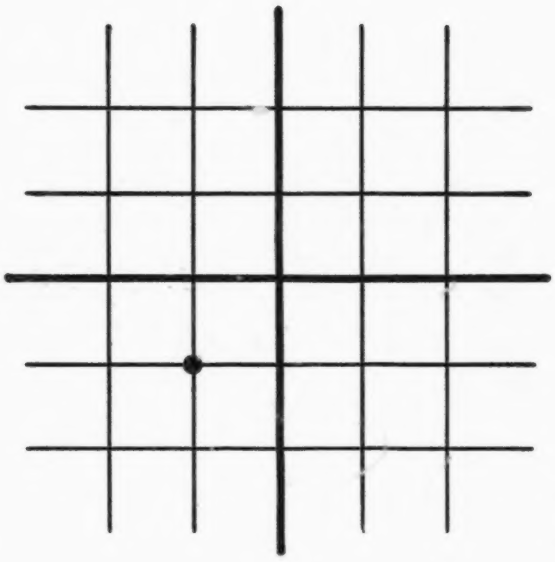
B. (⁻1,⁺1)

C. (⁻1,⁻1)

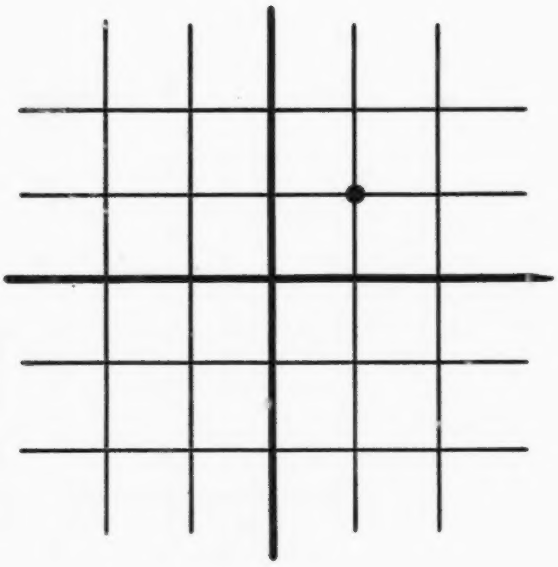
D. (⁺1,⁻1)

Problems 6, 7, 8, and 9 ask you to mark points on a graph.

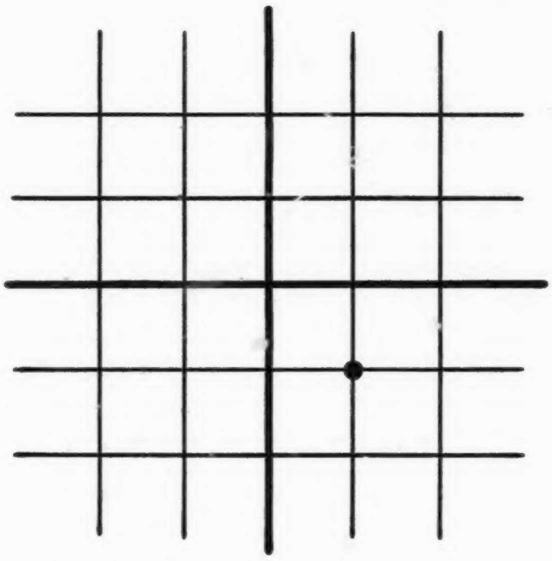
6. ($^-1$,$^-1$)



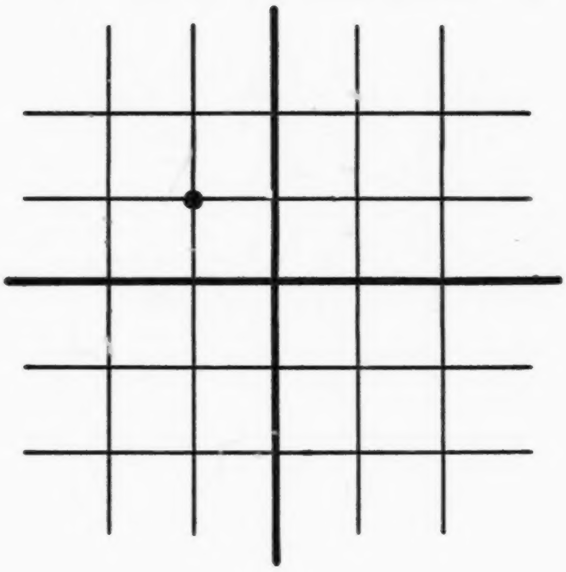
7. ($^+1$,$^+1$)



8. ($^+1$,$^-1$)



9. ($^-1$,$^+1$)



Agenda Item 10. What Can I take Back to My Class This Week?

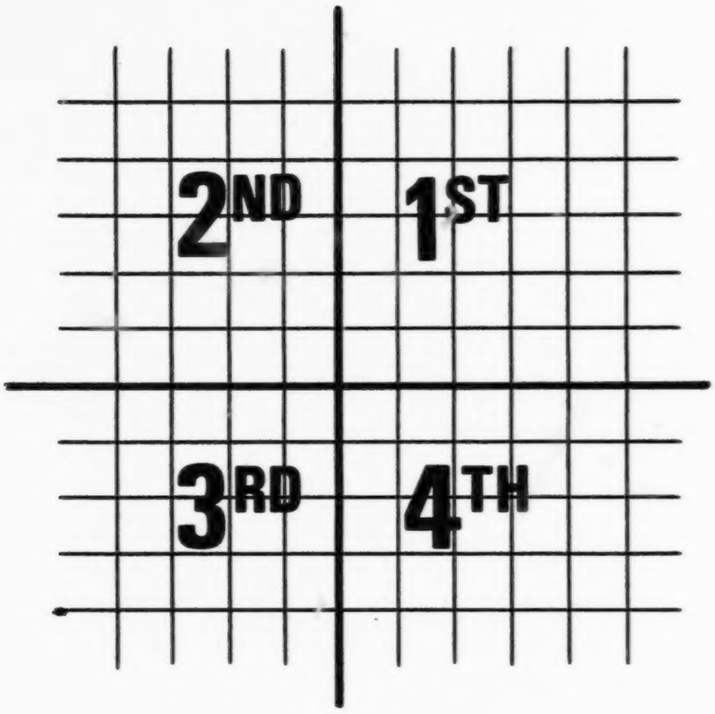
No doubt, many ideas have presented themselves which you would like to relay to your classes. If they seem appropriate, try them.

You should try to become familiar with the sentences -- true, false, and open. Remember, all need not be equations, e.g., $4 < \square < 10$.

Also try some work with open sentences with two variables. Make up your own if you like. Match this work to your needs.

If you graph the truth sets of some linear equations with your students, they will probably notice that the Tic Tac Toe game you play with them is really played on the Cartesian Plane. A mathematician would say it had been played in the first quadrant.

Notice how the axes naturally separate the plane into four quadrants. They are normally named as shown.



Tic Tac Toe is even more fun and useful using all four quadrants. Try it!

SUPPLEMENTARY MODERN MATHEMATICS

For Grades 1 - 9

An In-Service Course for Teachers

THIRD SESSION

Agenda:

1. Review of Linear Graphs
2. Film: "Experience with Linear Graphs"
3. Discussion

1. Review of Linear Graphs

In the Second Session, we considered equations of the general form

$$(1 \times \square) + 3 = \triangle$$

or

$$(2 \times \square) + 3 = \triangle ,$$

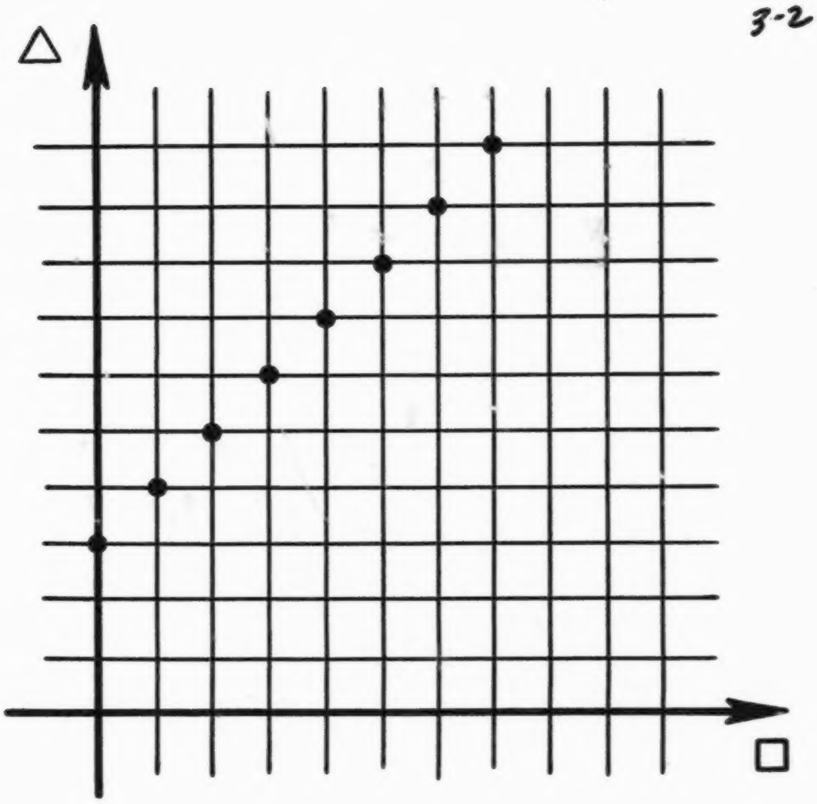
and so forth. The truth set for an equation can be represented by a table

$\square$	$\triangle$
0	3
1	4
2	5
3	6
4	7
.	.
.	.
.	.

Table for Truth Set of $(1 \times \square) + 3 = \triangle$

or by graph

3-2

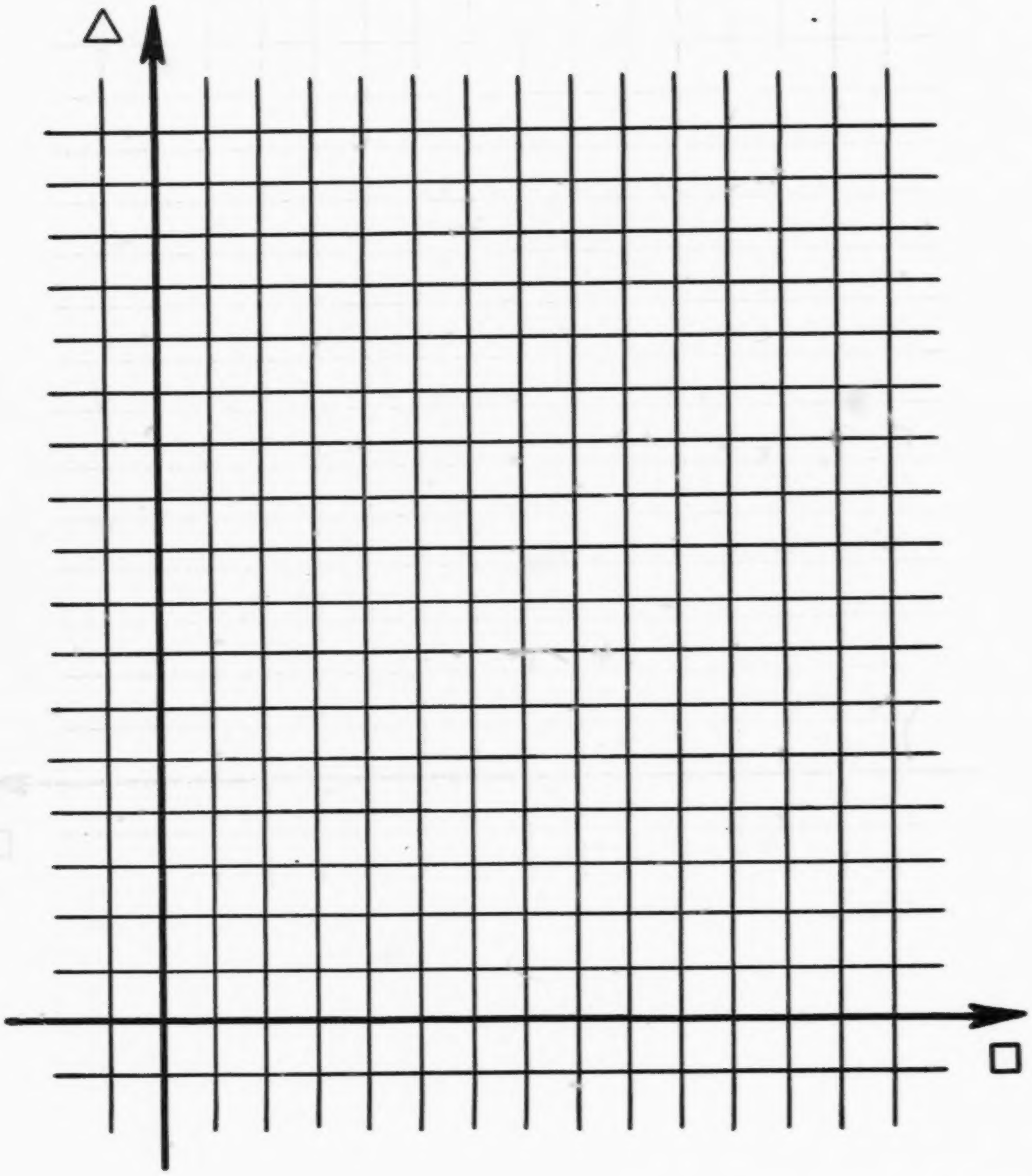


These graphs are not irregular; on the contrary, they exhibit certain very important patterns. The understanding of these patterns is fundamental to advanced work in mathematics. The discovery of these patterns is an exciting and rewarding experience for 4th, 5th, 6th, or 7th graders.

To make sure that you understand these patterns, try to solve the following problems:

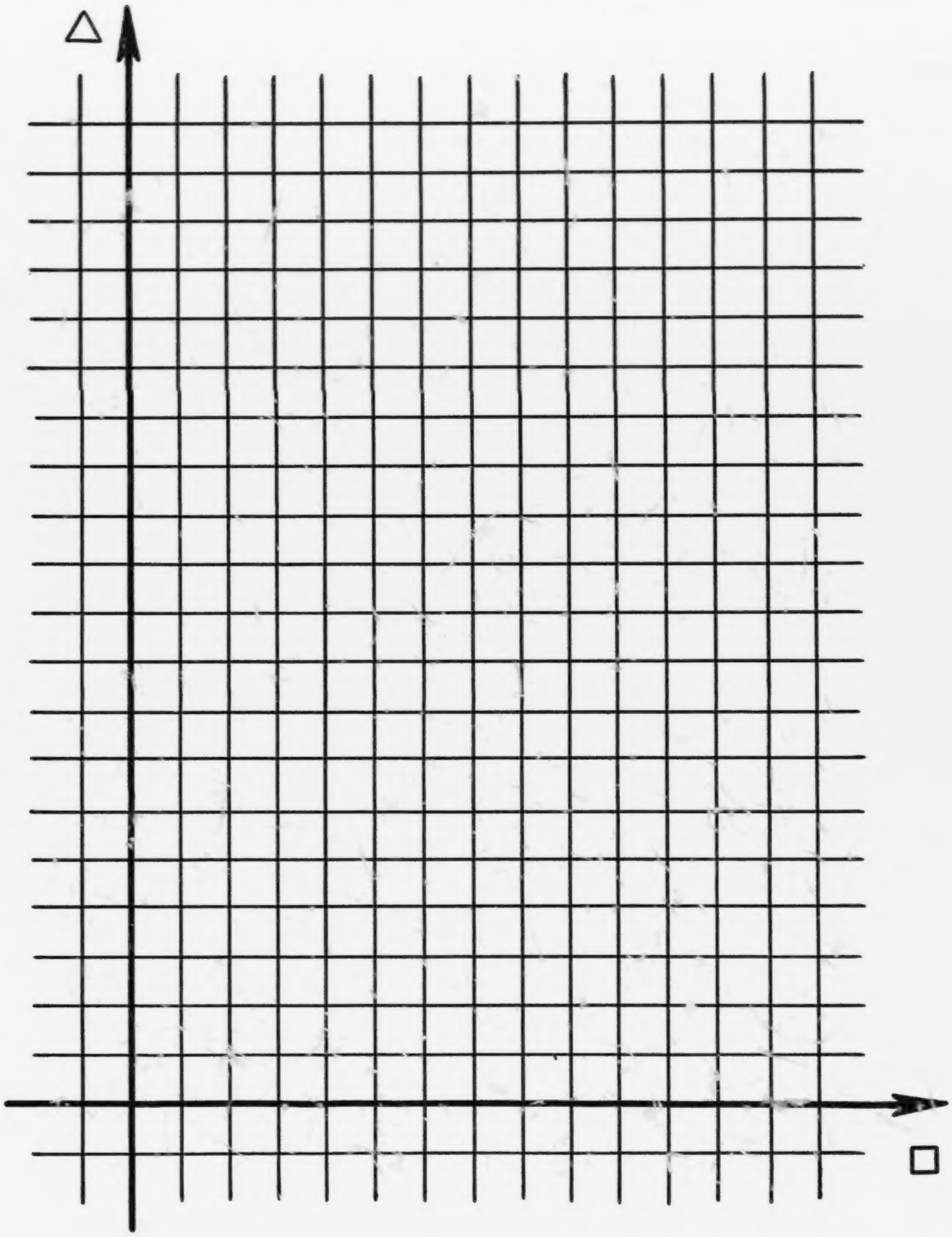
1. Make a graph for the truth set of

$$(2 \times \square) + 3 = \triangle .$$



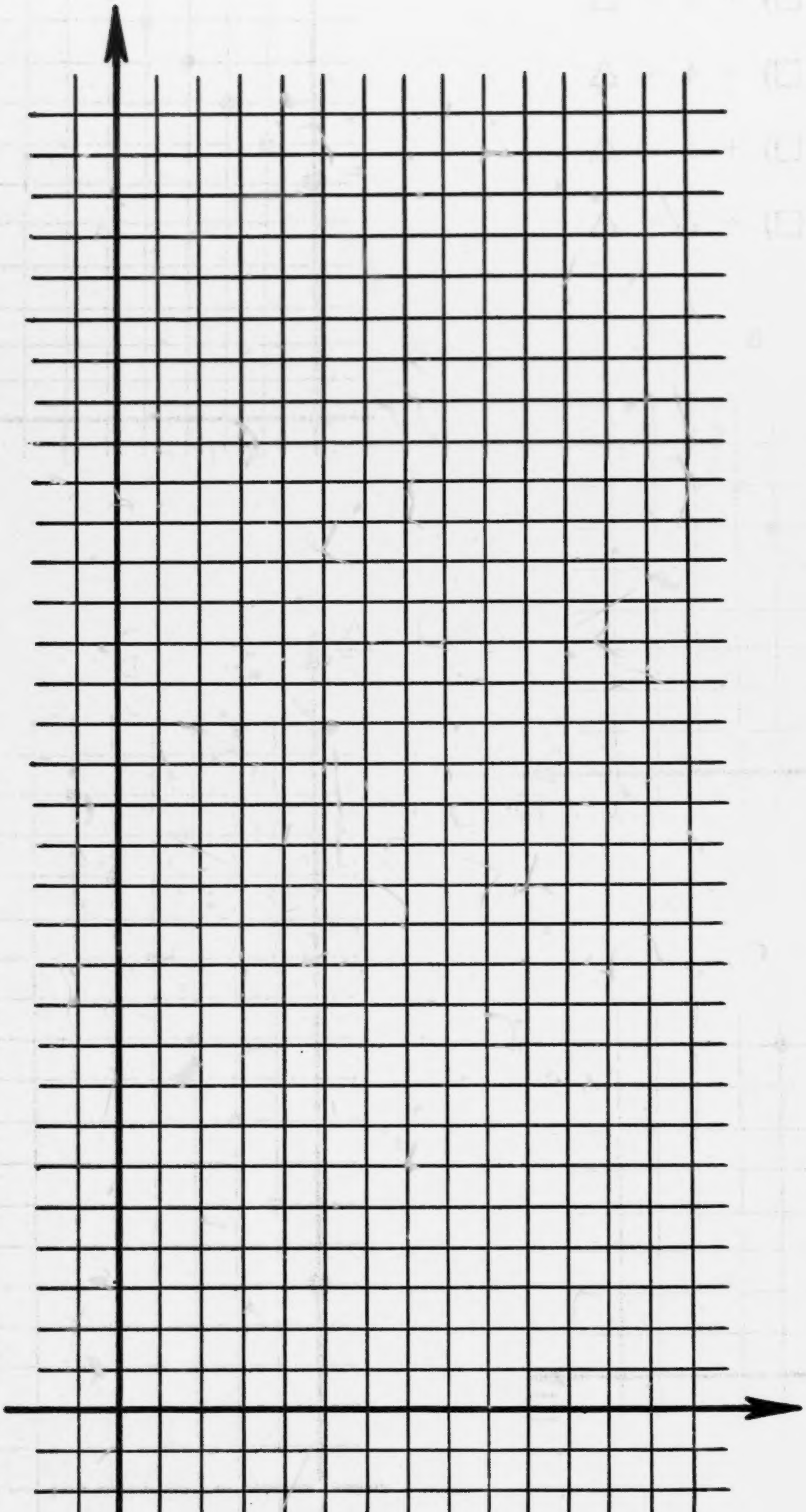
2. Make a graph for the truth set of

$$(3 \times \square) + 3 = \triangle .$$



3. Make a graph for the truth set of

$$(5 \times \square) + 3 = \triangle .$$



Can you match these equations and these graphs?

4. $(1 \times \square) + 4 = \triangle$

5. $(1 \times \square) + 5 = \triangle$

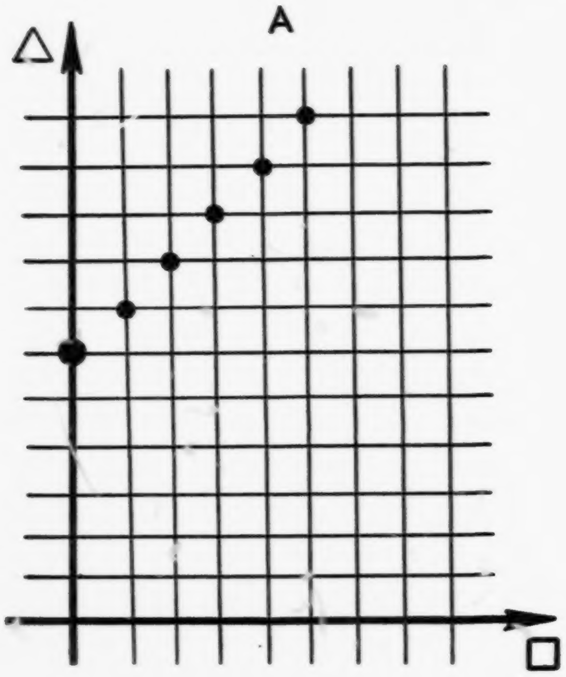
6. $(1 \times \square) + 6 = \triangle$

7. $(2 \times \square) + 4 = \triangle$

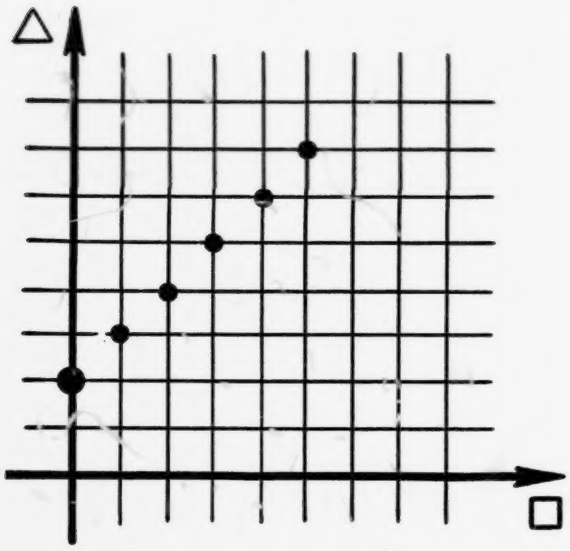
8. $(2 \times \square) + 5 = \triangle$

9. $(1 \times \square) + 2 = \triangle$

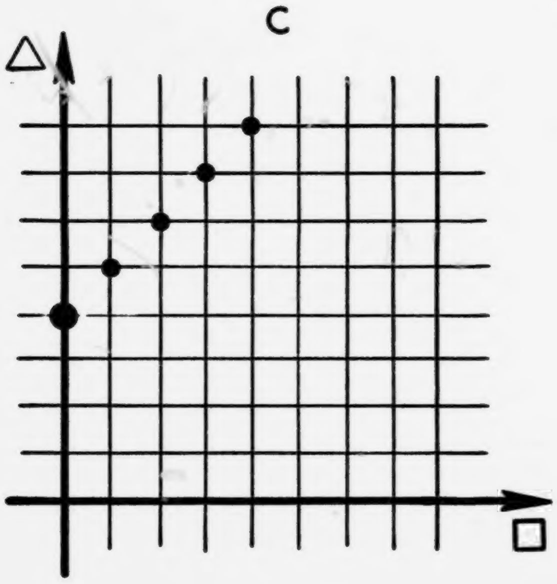
A



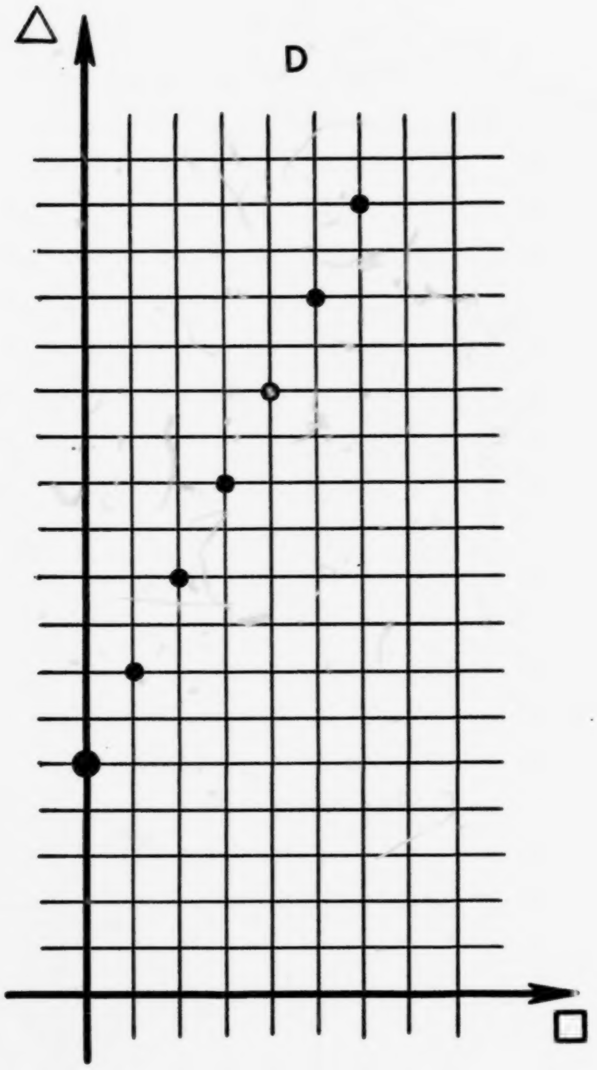
B



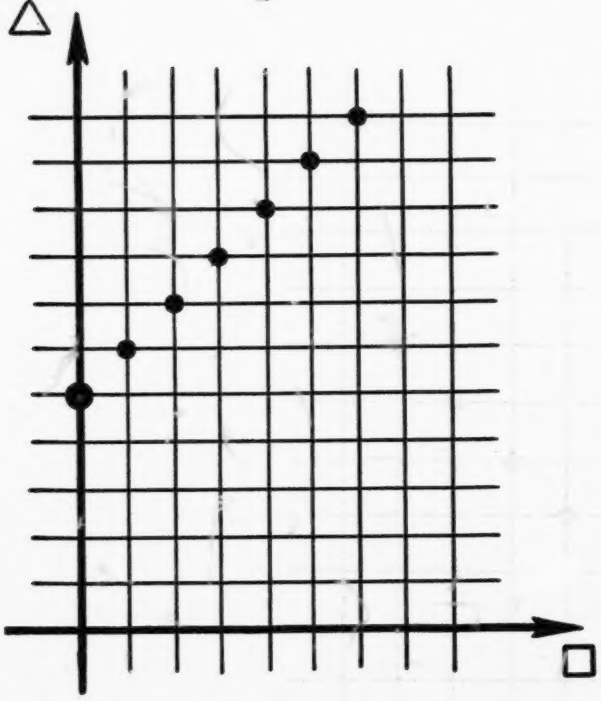
C



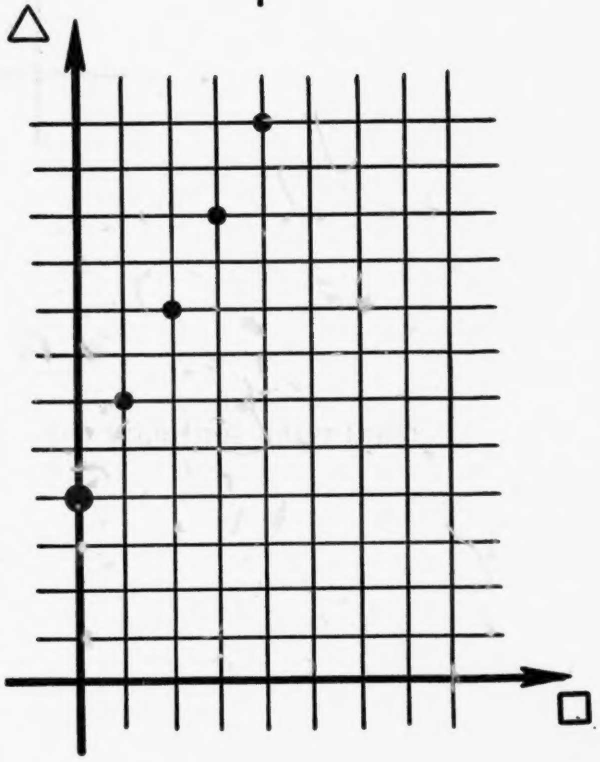
D



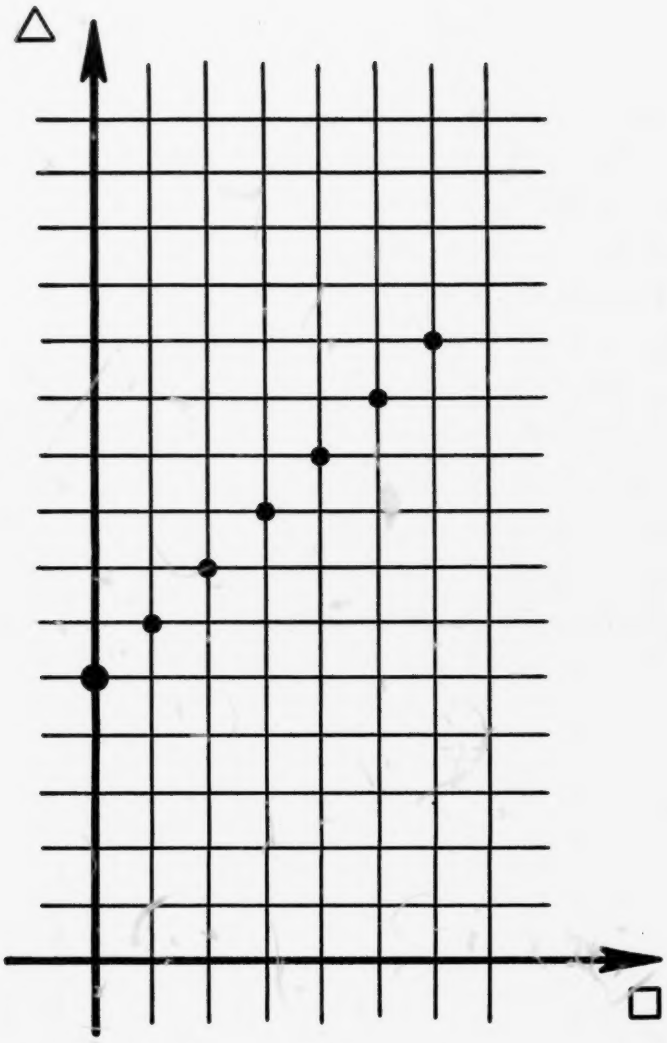
E



F



10. Cathy made this graph.

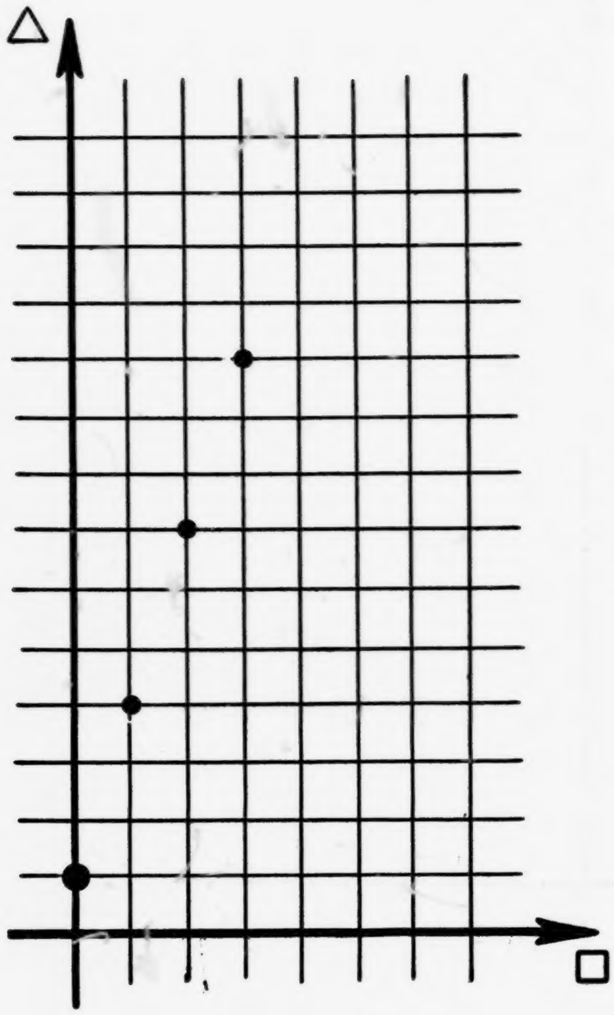


What open sentence was Cathy using?

(__ × □) + __ = △

↑ ↑

11. Jill made this graph.

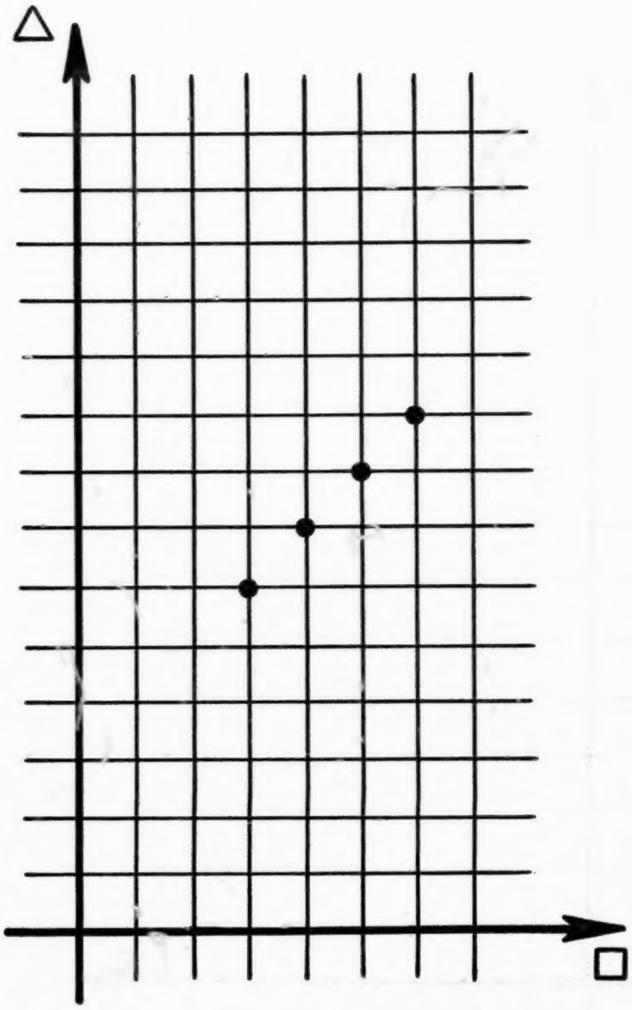


What open sentence was Jill using?

(__ × □) + __ = △

↑ ↑

12. Joe made this graph.



What equation was Joe using?

$$(\underset{\uparrow}{\underline{\quad}} \times \square) + \underset{\uparrow}{\underline{\quad}} = \triangle$$

YOU MAY WANT NOW TO VIEW THE FILM ENTITLED "Experience with Linear Graphs."

3. Discussion

By now there may be many things which you wish to discuss. We consequently leave this Agenda item open for your choice.

MADISON PROJECT

Syracuse University • Webster College

SUPPLEMENTARY MODERN MATHEMATICS

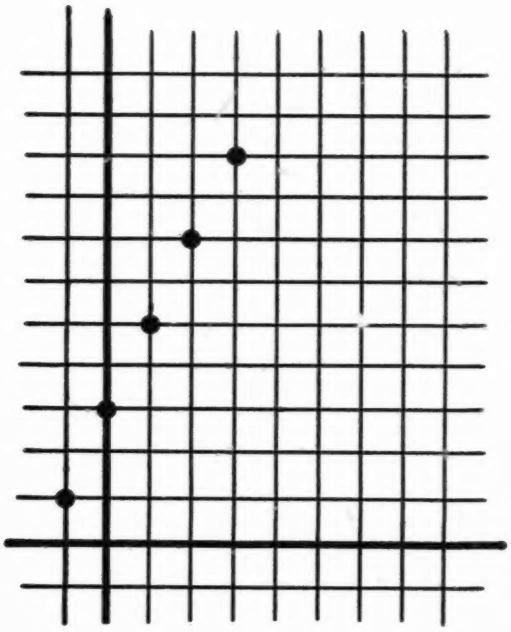
For Grades 1-9

REFERENCE BOOK

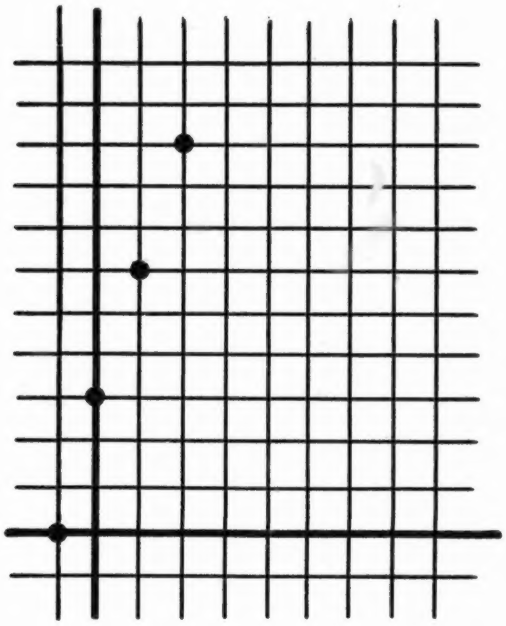
Third Session

Agenda: Item 1. Review of Linear Graphs

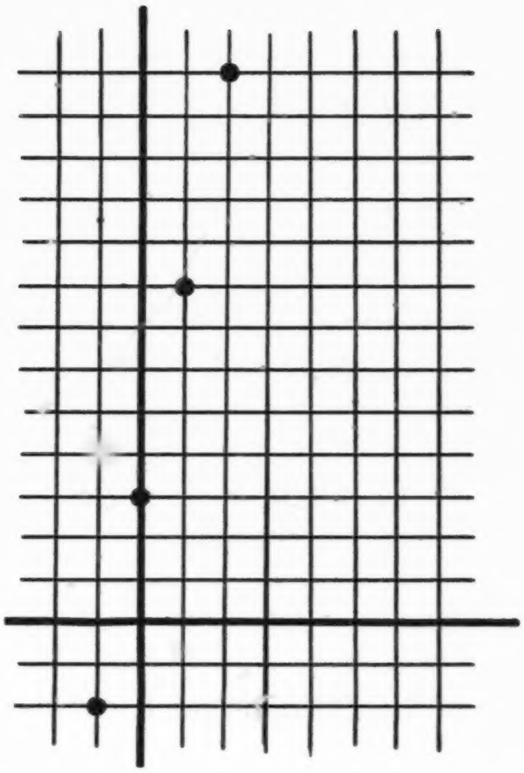
1. A graph of the truth set of (2 × □) + 3 = △



2. A graph of the truth set of (3 × □) + 3 = △



3. A graph of $(5 \times \square) + 3 = \triangle$



Equations 4 – 9 are correctly matched to graphs A – G as follows:

4. C	7. F
5. E	8. D
6. A	9. B

10. In deciding how to describe Cathy's graph, one should take advantage of the information given him.

Where does the graph intersect the $\triangle$ axis? What pattern is displayed in the points plotted in the graph? When working from a given open sentence and trying to plot points, what caught our attention?

When we notice that Cathy's graph can give us a pattern, (over one to the right and up one), and that the △ intercept is (0,5), we decide that the open sentence sought is

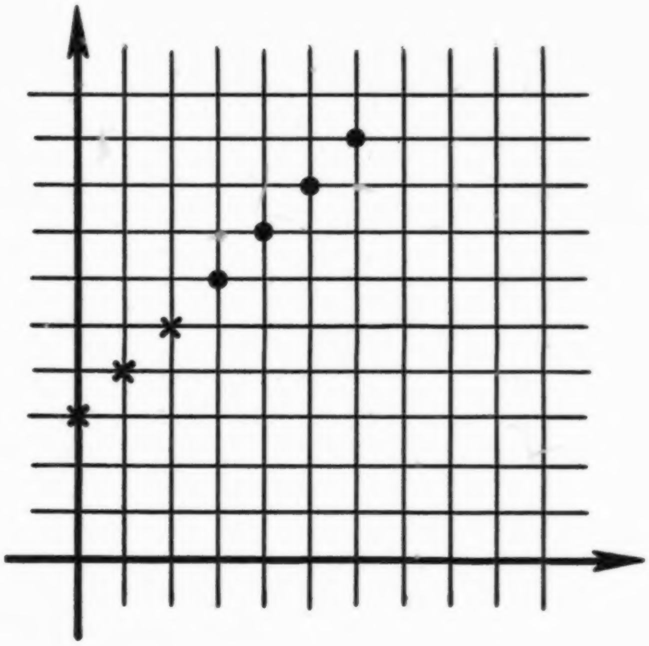
$$(\underline{1} \times \square) + \underline{5} = \triangle$$

11. Jill's graph intersects the △ axis at $^{+}1$ and has a pattern of one to the right and up three.

$$(\underline{3} \times \square) + \underline{1} = \triangle$$

12. Joe's graph. Since the △ intercept is not shown, we have to find it before writing the equation.

We do this by extending the pattern as far as necessary, like this:



It is now clear that the △-intercept is $(0, ^{+}3)$, and since the "pattern" is "over one to the right, and up one" (which means that the slope is $^{+}1$), the equation must be

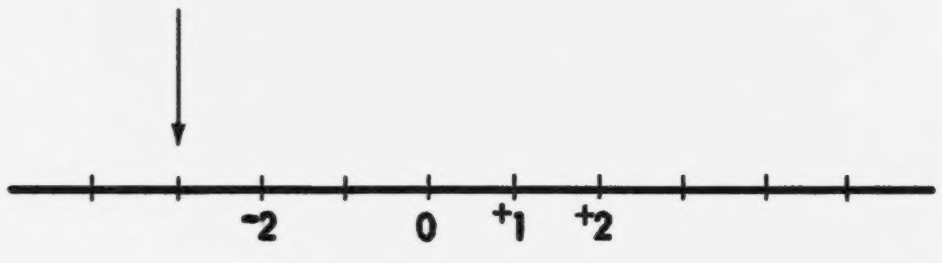
$$(^{+}1 \times \square) + {}^{+}3 = \triangle$$

Agenda: Item 3. Discussion

It probably is good to mention at this point that equations of the form $\triangle = (\_\_ \times \square) + \_\_$ are of an unusually fruitful type and offer us a wealth of information not so readily available from others. It may be that the student exposed only to this "slope intercept" form of equations could get the idea that whenever one looks at any equation, the slope and vertical axis intercept are represented immediately in the coefficients of $\square$ and in the constant term. The truth can probably be brought home best by presenting an equation such as $(3 \times \square) + 12 = (6 \times \triangle)$ casually and asking for predictions of the slope and vertical intercept. Following this, if a partial truth set is found, you may open some revealing discussions.

Throughout these first three packages, vocabulary has been presented and used that has probably not been commonplace heretofore. In general it is not our desire to push the use of precise vocabulary and then introduce the ideas illustrating this vocabulary. Instead, it is usually more successful to introduce new terminology only as required to converse about the topic at hand. It also seems best not to insist upon the adoption of "standard" terminology immediately, but rather to use it yourself in the correct sense every time the occasion arises. A simple example of such use can be illustrated by the following typical excerpt from a lesson.

(On board)



Teacher: Can anyone tell me what I should put under the indicated spot on the number line?

Jim: Minus three!

Teacher: Jim says negative three. Is he right? Will you please come up and write it where it goes, Jim?

The teacher has listened to the child's obvious meaning, and has not rejected the doubtful use of "minus" in place of "negative." Whenever he can do so without insulting a child, the teacher can profitably use correct terminology himself, thereby accelerating its adoption in future discussions.

SUPPLEMENTARY MODERN MATHEMATICS
For Grades 1 - 9
An In-Service Course for Teachers

FOURTH SESSION

Agenda:

Practice with Variables, Open Sentences, and Signed Numbers by means of "Quadratic Equations"

1. **Explanation**
2. **Try It Yourself**
3. **Careful (But Rapid) Readiness Building: Open Sentences and Variables**
4. **First Film Excerpt: Rapid Readiness Building "With the Light Touch": Open Sentences and Variables**
5. **Readiness: Use of Parentheses**
6. **Second Film Excerpt: Use of Parentheses**
7. **Readiness: Rule for Substituting**
8. **Third Film Excerpt: Rule for Substituting**
9. **Readiness: Arithmetic of Signed Numbers via "Pet-Shop Stories"**
10. **Fourth Film Excerpt: "Pet-Shop Stories"**
11. **Practice with Variables, Open Sentences, and Signed Numbers by means of "Quadratic Equations"**
12. **Fifth Film Excerpt: Practice with Variables, etc., in the Context of "Quadratic Equations"**
13. **Discussion**

Practice with Variables, Open Sentences, and Signed Numbers by means of "Quadratic Equations"

1. Explanation

i) Traditional teaching of mathematics made extensive use of drill. Actually, the amount of repetitive drill that is required can be greatly reduced, and often eliminated entirely, by providing practice in previously learned techniques during the course of exciting explorations of new concepts or new situations. This approach is central to the present lesson.

ii) We want our students to get practice in three things: variables, open sentences and signed numbers. Before showing how we can provide this practice in the course of novel situations, let us review the three ideas involved.

iii) Open Sentences. We want to distinguish three types of statements or sentences in mathematics:

$3 + 7 = 10$ we call a true statement.

$2 \times 2 = 5$ we call a false statement.

$3 + \square = 5$ we call an open sentence.

We may substitute numbers into the $\square$ in the open sentence above. When we do so, we obtain either a false statement

$$3 + \boxed{3} = 5$$

or else a true statement

$$3 + \boxed{2} = 5.$$

The collection of those numbers which make the open sentence true we say forms the *truth set* for the open sentence.

Thus, for the open sentence

$$3 + \square = 5$$

the *truth set* is

$$\mathcal{T} = \{2\}\ .$$

It is traditional in mathematics to use braces -- -- when we wish to indicate a "set" or "collection." Of course, a "collection" containing merely a single member, as in the case above, may seem unexciting. Suppose, however, we were working with the open sentence

$$3 < \square < 8,$$

containing the symbol " $<$ " which means "is less than." For simplicity, let us agree *not* to use fractions or decimals, but to restrict our attention to *whole numbers* such as 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, ...

What would the truth set be in this case?

(If you give up, consult the reference book for our answer.)

iv) *Variables*. One of the most important ideas in all of mathematics is the concept of *variable*. If, for example, we have an infinite list of statements such as

$$1 + 0 = 1$$

$$2 + 0 = 2$$

$$3 + 0 = 3$$

$$4 + 0 = 4$$

$$5 + 0 = 5$$

$$\vdots$$

(as usual, the final three dots indicate a list which continues without stopping), we can express this list by means of one single statement, thanks to the notion of variable:

$$\square + 0 = \square \ .$$

In order for this concept to work effectively, we need several agreements and observations:

1) We agree that whatever number we put in the first "$\square$," we will put this same number into all other occurrences of "$\square$" in this same statement.

Thus, if we put 3 into the 1st $\square$:

$$\boxed{3} + 0 = \square$$

we must also put 3 into the 2nd occurrence of $\square$:

$$\boxed{3} + 0 = \boxed{3} \ .$$

As another example, had we put 8 into the 1st $\square$:

$$\boxed{8} + 0 = \square$$

we would have been required to put 8 into the 2nd $\square$:

$$\boxed{8} + 0 = \boxed{8} \ .$$

We shall refer to this requirement that "the same number go into all of the $\square$'s" as the "rule for substituting."

2)[1] We also need an agreement concerning which numbers we are allowed to substitute into the $\square$. We can use the letter $\mathcal{A}$ (for "allowed") to indicate the collection of numbers that we are allowed to use.

---

[1] This section, numbered 2, deals with the collection $\mathcal{A}$ of numbers that we agree to allow for use in substituting into the $\square$. This question may seem unnecessarily complicated. Omit this section if you prefer -- it isn't essential at this stage of our work.

If we want to reproduce the list

$$1 + 0 = 1$$

$$2 + 0 = 2$$

$$3 + 0 = 3$$

$$4 + 0 = 4$$

$$\vdots \quad ,$$

then we write

$$\square + 0 = \square \ ,$$

and we write

$$\mathcal{A} = \{1, 2, 3, 4, 5, 6, 7, \ldots\} .$$

This represents the entire infinite list, as indicated by the use of three final dots.

Suppose we wanted to represent a much shorter list, containing only 3 statements

$$5 + 0 = 5$$

$$12 + 0 = 12$$

$$29 + 0 = 29.$$

We could do this by writing

$$\square + 0 = \square \ ,$$

with $\mathcal{A}$ given as

$$\mathcal{A} = \{5, 12, 29\} .$$

In section iii, on "Open Sentences," we considered the open sentence

$$3 < \square < 8.$$

Now, this problem would be very complicated if we used fractions and decimals, along with whole numbers. In order to keep it simple, we agreed to use only whole numbers. We could write this by saying:

$$\mathcal{A} = \{1, 2, 3, 4, 5, 6, \ldots\}.$$

Notice that we never put a number in the "truth set" $\mathcal{T}$ unless it is also included in the set $\mathcal{A}$. The mathematicians' language for expressing this is to say:

"the set $\mathcal{T}$ is, by definition, a subset of the set $\mathcal{A}$."

In most cases, it seems reasonable to use any numbers that you know -- 1, 2, 3, $\frac{1}{2}$, $\frac{7}{8}$, $^-1$, $^-21$, π, $\sqrt{2}$, and so on -- for substituting into the $\square$. Whenever we do not specify the set of "allowed" numbers, $\mathcal{A}$, then we mean that you are allowed to use any kind of number that you know about.

3) Remember, the legality of a substitution is a separate question from the truth or falsity of the resulting statement. There are four possible situations:

	legality	truth value
$\boxed{3} + \boxed{3} + \boxed{3} = 9$	legal	true
$\boxed{4} + \boxed{4} + \boxed{4} = 9$	legal	false
$\boxed{2} + \boxed{3} + \boxed{4} = 9$	illegal	true
$\boxed{3} + \boxed{4} + \boxed{5} = 9$	illegal	false

For Practice -- You Do It!

Fill in numbers for the following table:

	legality	truth value
$\square \times \square = 36$	legal	true
$\square \times \square = 36$	legal	false
$\square \times \square = 36$	illegal	true
$\square \times \square = 36$	illegal	false

In finding the truth set for an open sentence, we are of course restricting ourselves to legal substitutions which produce true statements. Thus, for the open sentence

$$\square + \square + \square = 9$$

the truth set is

$$\mathcal{T} = \{3\} .$$

v) Signed Numbers. Using the "pebbles-in-the-bag" approach, we have been able to take unsigned "counting numbers" 1, 2, 3, 4, ... and add or subtract them, expressing the answer as a signed number that tells us whether there are more or less pebbles in the bag than there were "when Joan (or whoever) said 'Go!'"

For example,

$$8 - 10$$

means "put 8 pebbles into the bag, and take 10 out."[1]

As a result of doing this, there are two pebbles less in the bag than there were when Joan said "Go!" We express this state of affairs by writing:

$$8 - 1 = {}^{-}2$$

---

[1] Recall that there were already many pebbles in the bag "when Joan said 'Go!'"

(read: "eight minus ten equals negative two").

As a second example,

$$16 - 20 + 6$$

would mean "put 16 pebbles into the bag, take 20 out, then put 6 in." We would have

$$16 - 20 + 6 = {}^{+}2.$$

QUESTION: What would this come out to be:

$$20 - 25 + 3 = ?$$

vi) As mentioned earlier, we want our students to get practice in the concepts of variable, open sentence, and signed numbers.

We do not want to resort to repetitious drill.

We want this practice to appear in the course of exciting explorations of some new topic.

We also want the student to get in the habit of looking for useful patterns in his mathematical work. Consequently, we try to arrange our materials so that there are many interesting and helpful "patterns" lurking just beneath the surface, which the student can discover if he is sufficiently alert.

For all of these reasons, we have chosen a rather special kind of open sentence for use in the following exercises. What we are using are known technically as "quadratic equations written in standard form," but this designation is not important at this stage.

In the next section, you can practice with some of these equations yourself.

2. Try It Yourself!

1. For the open sentence

$$(\square \times \square) - (5 \times \square) + 6 = 0$$

the truth set contains two numbers, each of which is an ordinary "counting number" (like 1, 2, 3, 4, 5, ...). Can you find either or both of them?

Suggestion: Proceed by trial-and-error. For example, try 1:

$$(\boxed{1} \times \boxed{1}) - (5 \times \boxed{1}) + 6 = 0$$

$$1 \quad - \quad 5 \quad + 6 = 0$$

But: $1 - 5 + 6 = {}^{+}2$, as we can see from thinking about the "pebbles-in-the-bag" idea.

Consequently,

$1 - 5 + 6 = 0$ is false,

and 1 is not a number in the truth set.

Try some other numbers.

For $(\square \times \square) - (5 \times \square) + 6 = 0$,

the truth set $\mathcal{T}$ is $\{\ \ ,\ \ \}$.

2. Can you find the truth set for this open sentence:

$$(\square \times \square) - (12 \times \square) + 35 = 0$$

$$\mathcal{T} = \{\ \ ,\ \ \}$$

Two numbers, both positive integers.

3. **Can you find the truth set for this open sentence?**

$$(\square \times \square) - (8 \times \square) + 15 = 0$$

$$\mathcal{T} = \{\ ,\ \}$$

4. **Have you found the "secret method" for finding these truth sets? If not, you can still find $\mathcal{T}$ just the same, by using "trial and error."**

5. **Can you find the truth set for this open sentence?**

$$(\square \times \square) - (12 \times \square) + 20 = 0$$

$$\mathcal{T} = \{\ ,\ \}$$

6. **Can you find the truth set for this open sentence?**

$$(\square \times \square) - (9 \times \square) + 20 = 0$$

$$\mathcal{T} = \{\ ,\ \}$$

3. Careful (But Rapid) Readiness Building: Open Sentences and Variables.

The title above largely speaks for itself. We want to build readiness for every foreseeable complication, but we want to do it quickly and "with the light touch." It would be hard to say which is worse: no readiness building, or heavy-handed slow laborious tedious readiness building.

To see how "rapid readiness building with the light touch" can be used in relation to the concepts of open sentences and variables, view Film Excerpt Number 1.

5. Readiness: Use of Parentheses

Here, again, we are concerned with "rapid readiness building with the light touch." Cf. Film Excerpt #2.

7. Readiness: Rule for Substituting

Once again, we seek careful, easy, quick preparation for an important idea.

Cf. Film Excerpt #3.

9. Readiness: Arithmetic of Signed Numbers via "Pet-Shop Stories"

"Pet-shop stories" provide an alternative to the "Pebbles-in-the-Bag" model for problems like

$$5 - 7 = {}^{-}2$$

$$5 + 3 = {}^{+}8$$

$$10 - 12 + 3 = {}^{+}1$$

etc.

Cf. Fourth Film Excerpt.

11. Practice with Variables, Open Sentences, and Signed Numbers by means of "Quadratic Equations"

Now we are ready to put all of these pieces into an appropriate total context.

Cf. Film Excerpt #5.

13. Discussion

You probably have plenty of questions and remarks of your own. Here are some that frequently arise:

1. The children in the film are in grades 3 through 7. Ruth is a 3rd grader.
2. What about the children who don't discover the secret?
3. Why did we have the children working with quadratic equations? What purpose did we have in mind?
4. Isn't this too abstract for children this age?

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SUPPLEMENTARY MODERN MATHEMATICS

For Grades 1-9

REFERENCE BOOK

Fourth Session

Agenda Item 1. Explanation

iii) $3 < \square < 8$

One could read the above open sentence as: 3 is less than $\square$, and $\square$ is less than 8. For our truth set $\mathcal{T}$, we want in this case all whole numbers that are greater than 3 and yet smaller than 8. The only whole numbers satisfying these criteria are 4, 5, 6, and 7, and we write $\mathcal{T} = \{4, 5, 6, 7\}$.

v) $20 - 25 + 3 = \underline{^{-}2}$

To really convince ourselves that this is true, we can mentally pose a "pebbles-in-the-bag" situation using the above numbers.

Agenda Item 2. "Try it Yourself"

1. The truth set $\mathscr{T}$ for $(\square \times \square) - (5 \times \square) + 6 = 0$ is $\{2,3\}$. We can be reasonably sure of this since we observe that "$(\boxed{2} \times \boxed{2}) - (5 \times \boxed{2}) + 6 = 0$" and "$\boxed{3} \times \boxed{3} - (5 \times \boxed{3}) + 6 = 0$" are each true statements and that "$\square \times \square - (5 \times \square) + 6 = 0$" becomes a false statement when any other number is used in the $\square$.

2. For $(\square \times \square) - (12 \times \square) + 35 = 0$, $\mathscr{T} = \{5,7\}$.

3. For $(\square \times \square) - (8 \times \square) + 15 = 0$, $\mathscr{T} = \{5,3\}$.

5. For $(\square \times \square) - (12 \times \square) + 20 = 0$, $\mathscr{T} = \{10,2\}$.

6. For $(\square \times \square) - (9 \times \square) + 20 = 0$, $\mathscr{T} = \{4,5\}$.

Agenda Item 13. Discussion

2. What about the children who don't discover the secret? Opinions vary here, but eventually it is usually shown that the additional incentive to those not having discovered the "secrets" keeps interest hones. The pride involved in knowing the "secrets" when others don't know them also makes this topic and "secret" approach worthy of use. Also, for many students this may be the first time that they actually see a small piece of mathematics discovered rather than authoritatively stated.

3. Why did we have children working with quadratic equations? This really has been discussed at some length in section vi. In any event, we must remember that our purpose is not to have children become proficient in solving or manipulating quadratic equations.

4. Isn't this too abstract for children this age? Experience with many children has shown that this is not the case and that children profit from and enjoy working with this topic.

SUPPLEMENTARY MODERN MATHEMATICS

For Grades 1 - 9

An In-Service Course for Teachers

FIFTH SESSION

Agenda:

1. "Make Up a Rule ..."
2. Preliminary Notes on Film Excerpt
3. First Film Excerpt: "Make Up a Rule ..."
4. General Review and Discussion

1. "Make Up a Rule ..."

The idea for this lesson was suggested by Professor W. Warwick Sawyer of Wesleyan University, Middletown, Connecticut.

1. Mary made up a rule. When Jeff told her "3," she answered "5." When Jeff told her "0," she answered "2."

What do you think Mary answered when Jeff said "11"?

2. What rule was Mary using? Can you write it, using □ to represent numbers Jeff says, and △ to represent Mary's answers?

Answer: □ + 2 = △.

3. Jill made up this rule. When Tony said "0," Jill answered "3." When Tony said "1," Jill answered "5."

What do you think Jill answered when Tony said "2"?

4. What rule was Jill using? Can you write it, using the □ and △ notation?

Answer: ________________

Why don't you discuss this idea, and practice using it? When you are ready, go to Agenda Item Number Two.

2. Preliminary Notes on Film Excerpt

1. Normally it is desirable to let the children make up their own rules. One difficulty can arise, however: some children make rules so somplex that no other children can guess them. In order to avoid this while filming a lesson, a number of possible "rules" were written on cards, and children chose among these.

2. Several children worked together on each "rule," in order to minimize confusion caused by arithmetical errors. If you try this in your own classes, you will speedily find these various complications, and will develop methods for dealing with them.

3. In the film excerpt, film viewers can see the "rule" that the children are using. This is achieved as in the television program "What's My Line" -- the film viewers can see the equation the children are using, but *the other children in the class cannot*. The other children in class have the task of *guessing* what rule the "panelists" are using. *The card which can be seen on the table in class was left over from the previous problem, and does not relate to the problem presently under discussion*.

YOU MAY NOW WANT TO VIEW THE FIRST FILM EXCERPT.

4. General Review and Discussion

1. You may find it valuable to try playing the "Make Up a Rule ..." game some more yourselves.

2. In trying to guess rules, it is sometimes valuable to make a graph in the usual way, representing the "□ " number along a horizontal axis, and representing the "△" number along a vertical axis.

3. Usually a child "guesses" a rule in words, for example by saying "you double the number and then subtract one." We then see if he can write it in □, △ notation (in this case, as $(\square \times 2) - 1 = \triangle$). Whichever child first writes the rule correctly in □, △ notation is allowed to select one or two other children to serve with him as the "panelists" for the next problem.

4. Sometimes a child will say he is using the rule

$$(\square + 3) \times 2 = \triangle.$$

He will produce a table like this:

□	△
0	6
1	8
2	10
3	12
4	14
100	206
1,000	2,006

Some other child may "guess" this in the form

$$(2 \times \square) + 6 = \triangle.$$

Now the question arises: should this guess count as correct, or not?

The mathematical decision on this, which we shall consider further next week, is that, since both rules will always produce the same "answers" when we substitute numbers into the "$\square$," both MUST BE CONSIDERED CORRECT. (This brings up the difference between "formula" and "function," which we shall consider in the next session.)

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SUPPLEMENTARY MODERN MATHEMATICS

For Grades 1-9

REFERENCE BOOK

Fifth Session

Agenda Item 1. "Make Up a Rule"

1. Mary's rule seems straightforward. When Jeff told her "3," she answered "5." When Jeff told her "0," she answered "2."

In each case Mary responded with a unique number related in some way to the number given her. We note that the number she gives back in both cases is larger by two than the number given her. Under this rule, Mary would give back the number which is 2 larger than 11 when given 11, i.e., 13.

3. Jill's rule, though not quite so obvious, does again assign a unique number to each number offered; we may be able to discover the pattern used if we are persistent.

A first guess might be that Jill is adding 3. This fails when we remember that Jill said "5" when given "1."

After several tries, we notice that when a number is offered and a reply given, several "rules" may seem, each in turn, able to be used. As additional numbers and replies are examined, our choice of rules is narrowed. In the case of Jill's rule, we would eventually be prompted to notice that the rule which takes the number 0 and yields 3, takes 1 and yields 5, would take 2 and yield 7; i.e., takes the number, doubles it, then adds 3.

In box, triangle ($\square$, $\triangle$) notation, we could write $(2 \times \square) + 3 = \triangle$.

This game of "Make Up a Rule" is very versatile. It has been used from grades one on up. The rules can be made in any desired degree of difficulty suiting the ability of the group.

SUPPLEMENTARY MODERN MATHEMATICS

For Grades 1 - 9

An In-Service Course for Teachers

SIXTH SESSION

Agenda:

1. Review (Optional): "Make Up a Rule ..."
2. "Formulas" vs. "Functions"
3. First Film Excerpt
4. Discussion
5. Addition of Signed Numbers, Using the Number Line
6. Postman Stories for Addition
7. Second Film Excerpt: Postman Stories for Addition
8. Discussion

1. Review (Optional): "Make Up a Rule ..."

If you wish, have 2 or 3 teachers serve as a "panel" to "make up a rule" -- as in last week's film -- and see if you can guess their rule. If you guess it in words, you must then write it in □,△ notation in order to be declared winner. Whoever first writes the rule correctly in □,△ notation selects the panelists for the next problem,

An example might go like this:

Mary, John, and Kris have made up a rule.

The class asks Mary, John, and Kris to use their rule on 0, and tell the answer.

Mary, John, and Kris answer: $^{-}1$.

We can enter this on a table, as follows:

□	△
0	$^{-}1$

Here, we use "□" to denote the numbers the class asks them to use, and "△" to indicate Mary, John, and Kris's answer.

The class now asks them to use 1, 2, 3, and 10. Here are their answers:

□	△
0	$^{-}1$
1	1
2	3
3	5
10	19

At this point, suppose Francis says: "I know your rule: you double it and subtract one!"

Francis, in order to be declared winner, must come to the board and write this in □, △ notation. If, for example, he writes

$$(\square \times 2) - 1 = \triangle,$$

he is right, and he now chooses the team to make up the next "rule."

WHY DON'T YOU TRY THIS YOURSELF, AT THIS POINT?

2. "Formulas" vs. "Functions"

You may have already encountered the distinction between "formula" and "function." This is a fundamental distinction for future work in mathematics. That it appears naturally in the children's work here is fortunate, and we want to make the most of this opportunity.

Here is how it works:

Suppose, in the Mary, John, and Kris rule we just looked at, that Francis had written

$$(\square \times 2) - 1 = \triangle.$$

Suppose Mary, John, and Kris rejected this, saying: "No! You're wrong! Our rule was

$$(2 \times \square) - 1 = \triangle.$$

How do we settle this argument?

Clearly, the printed expression

$$(\square \times 2) - 1 = \triangle$$

is not the same as the printed expression

$$(2 \times \square) - 1 = \triangle.$$

As far as printed expressions go, Mary, John, and Kris are right. We can express this by saying that these two formulas are different.

Francis has not guessed the correct formula!

However, we shall make the ruling that Francis is not required to guess the same formula that Mary, John, and Kris are using.

Now -- what was correct in Francis' guess? The answer is that he has guessed a formula which will produce the correct table. That is, if we use the formula

$$(\square \times 2) - 1 = \triangle,$$

we get the table

$\square$	$\triangle$
0	$^{-}1$
1	1
2	3
3	5
4	7
5	9
.	.
.	.
.	.

which is exactly the same table that we would get if we used instead the formula

$$(2 \times \square) - 1 = \triangle.$$

When we wish to focus attention on the resulting table of numbers, rather than on the "formula," we speak of a function.

Thus, Francis has written the correct function, even though (as is permissable) he has represented this function by means of a different formula.

The final verdict: Francis is right, and he now picks the team for the next problem.

Since a table

$\square$	$\triangle$
0	$^{-}1$
1	1
2	3
3	5
4	7
5	9
⋮	⋮

is a listing of ordered pairs of numbers, you can see why some "modern" books refer to a function as a set of ordered pairs of numbers.

Exercises for practice:

1. Do these 2 different formulas represent the same function, or not?

$$\square + \square = \triangle$$

$$2 \times \square = \triangle$$

2. Do these 2 different formulas represent the same function, or not?

$$(2 \times \square) + (3 \times \square) = \triangle$$

$$5 \times \square = \triangle$$

3. Do these 2 different formulas represent the same function, or not?

$$(2 \times \square) + (4 \times \square) = \triangle$$

$$(6 \times \square) + \square = \triangle$$

4. Do these 2 different formulas represent the same function, or not?

$$\square - 7 = \triangle$$

$$\square + {}^{-}7 = \triangle$$

YOU MAY NOW WISH TO VIEW THE FIRST FILM EXCERPT.

4. Discussion

1. Is $\square - 7 = \triangle$ the same formula as $\square + {}^{-}7 = \triangle$?

2. Does $\square - 7 = \triangle$ represent the same function as $\square + {}^{-}7 = \triangle$?

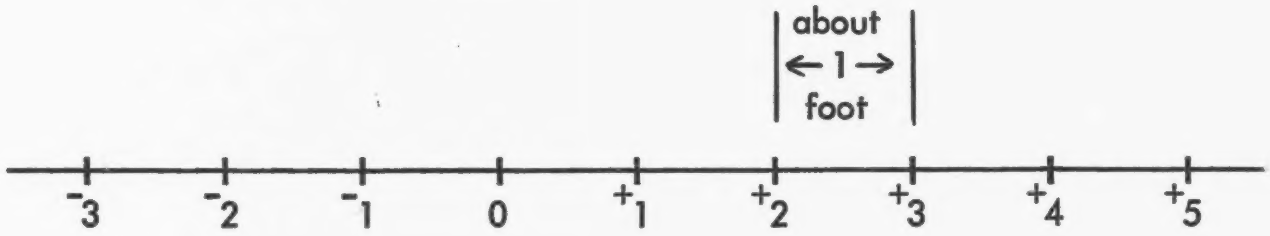
3. In order to "win" in guessing the rule, is it necessary to guess the correct formula, or the correct function? Which is a more reasonable task?

4. At what grade level could you use this material?

5. Have you tried this "Make Up a Rule..." game in your own class? How did it work? What kinds of difficulties arose?

5. Addition of Signed Numbers, Using the Number Line

For young children, it seems desirable to mark a number line on the floor, with about a foot between consecutive integer marks:



The children can stand on this, and can walk, hop, or jump along it to represent addition and subtraction.

The addition

$$^{+}3 + {}^{+}2$$

could be represented by a child standing on $^{+}3$, and hopping 2 units in the positive direction.

The addition

$$^{+}5 + {}^{-}2$$

could be represented by a child standing on $^{+}5$, and hopping 2 units in the negative direction.

The addition

$$^{-}3 + {}^{-}4$$

could be represented by a child standing on $^{-}3$, and then hopping 4 units in the negative direction.

YOU DO IT: Why don't you mark a number line on the floor, and use it (as in the examples above) to represent

a) $^{-}1 + {}^{-}1$

b) $^{-}3 + {}^{+}3$

c) $^{-}4 + {}^{+}5$

d) $^{+}1 + {}^{+}2$

e) $0 + {}^{-}3$

For older children, you may prefer to use this same approach with a number line drawn in chalk on the blackboard, simply indicating appropriate skips.

6. Postman Stories for Addition

We can also represent addition of signed numbers by "postman stories."

For

$$^{+}2 + {}^{+}3,$$

we pretend that $^{+}2$ represents a check for \$2, and that $^{+}3$ represents a check for \$3. Consequently,

$$^{+}2 + {}^{+}3$$

means that the postman comes and delivers two letters: one a check for \$2, and the other a check for \$3.

As a result of this visit, we are richer by an amount of \$5, so we can write

$$^{+}2 + {}^{+}3 = {}^{+}5.$$

Needless to say, it is well to verify that your students know what you mean by a check, and by a bill (as in the case of a bill from the department store, or a bill from the dentist, as opposed to currency).

For the addition problem

$$^{-}2 + {}^{-}5,$$

we pretend that the $^{-}2$ represents a bill for \$2, and the $^{-}5$ represents a bill for \$5. Consequently,

$$^{-}2 + {}^{-}5$$

can be interpreted as the postman bringing you two letters, one containing a bill for \$2, and one containing a bill for \$5.

As a result of this delivery, we are poorer by \$7, so we write

$$^{-}2 + {}^{-}5 = {}^{-}7.$$

(Question: how would you represent $^{-}2 + {}^{-}5$ on the number line? Would you get the same answer?)

Questions for Practice: Can you make up a postman story to match each of these additions?

a) $^{+}5 + {}^{-}3$

b) $^{+}2 + {}^{+}7$

c) $^{-}1 + {}^{-}5$

d) $^{-}8 + {}^{+}8$

e) $^{+}7 + 0$

YOU MAY NOW WANT TO VIEW THE SECOND FILM EXCERPT.

8. Discussion

We leave this open for your own questions and remarks.

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SUPPLEMENTARY MODERN MATHEMATICS

For Grades 1-9

REFERENCE BOOK

Sixth Session

Agenda: Item 2. "Formulas" vs. "Functions"

Exercises for practice:

1. □ + □ = △

   2 x □ = △

To decide whether or not these formulas represent the same function, we should use each formula in turn to produce a table of truth values; i.e., a partial truth set for each. It must by necessity be partial since the possible list of ordered pairs that work is unending.

If the tables are identical in content, we will say that both tables do represent the same function.

Conversely, should we produce tables not identical (having an entry in one which is not possible in the other), then the formulas surely represent separate functions.

For the formula

□ + □ = △

we produce the table

□	$^{-}2$	$^{-}1$	0	1	2	3	4 . . .
△	$^{-}4$	$^{-}2$	0	2	4	6	8 .

For the formula

2 x □ = △

we produce the table

□	$^{-}3$	$^{-}2$	$^{-}1$	0	1	2	3	4 . . .
△	$^{-}6$	$^{-}4$	$^{-}2$	0	2	4	6	8 .

We notice that there is no entry (ordered pair of numbers) in the first table which is not, or cannot be, in the second, and vice versa. We therefore say that □ + □ = △ and 2 x □ = △ are formulas representing the same function.

2. $(2 \times \square) + (3 \times \square) = \triangle$

   $(5 \times \square) = \triangle$

From

$$(2 \times \square) + (3 \times \square) = \triangle$$

we construct this table

$\square$	$^-2$	$^-1$	0	1	2	3	...
$\triangle$	$^-10$	$^-5$	0	5	10	15	.

From

$$(5 \times \square) = \triangle$$

we construct this table.

$\square$	$^-2$	$^-1$	0	1	2	3	4	...
$\triangle$	$^-10$	$^-5$	0	5	10	15	20	.

Since the tables are alike we conclude

$$(2 \times \square) + (3 \times \square) = \triangle$$

and

$$(5 \times \square) = \triangle$$

represent the same function.

3. $(2 \times \square) + (4 \times \square) = \triangle$

   $(6 \times \square) + \square = \triangle$

From

$$(2 \times \square) + (4 \times \square) = \triangle$$

we can form this table

$\Box$	$^{-}1$	0	1	2	3	. . .
$\triangle$	$^{-}6$	0	6	12	18	.

From

$$(6 \times \Box) + \Box = \triangle$$

we can form this table

$\Box$	$^{-}1$	0	2	3	. . .
$\triangle$	$^{-}7$	0	14	21	.

The tables are not the same, therefore we conclude that the two formulas represent two distinct functions. This is probably a good time to note that while the tables as a whole are different, the pair 0,0 is in each. Had we considered abbreviated tables as shown below, we could easily have been misled.

$$(2 \times \Box) + (4 \times \Box) = \triangle$$

$\Box$	0	. . .
$\triangle$	0	

$$(6 \times \Box) + \Box = \triangle$$

$\Box$	0	. . .
$\triangle$	0	

With only this much information one might have concluded that both formulas represent the same function. It is not easy to say how many entries (ordered pairs) you should put in your tables and how many numbers you should check, since the complexity of the function involved affects this. Check enough to really convince yourself.

In a later lesson dealing with identities, axioms, and theorems we will consider other methods of deciding whether or not two formulas really represent the same function.

4. $\Box - 7 = \triangle$

$\Box + {}^{-}7 = \triangle$

Considering only the lessons to date, we really have no official method for deciding whether or not the two formulas above represent the same function. The reason is that we have not discussed the combining of signed numbers as is necessary in $\square + {}^{-}7 = \triangle$.

Agenda: Item 4. Discussion

1. Is

$$\square - 7 = \triangle$$

the same formula as

$$\square + {}^{-}7 = \triangle \ ?$$

No! If they were the same formula, we would expect one to be a rubber stamp copy of the other.

2.

$$\square - 7 = \triangle$$

represents the same function as

$$\square + {}^{-}7 = \triangle$$

if and only if their truth sets are identical; i.e., if there is no entry possible in the truth set of

$$\square - 7 = \triangle$$

that is not possible in

$$\square + {}^{-}7 = \triangle,$$

and vice versa. As we shall see in our subsequent work with signed numbers, these do represent the same function.

3. To "win" in guessing the rule, it is sufficient to guess the correct function. This is a much more reasonable task than to be asked to guess the exact formula being used, which would require you to read the other person's mind.

To show just how tough finding the exact formula could be, we consider the following list of formulas and note that any and all of them represent the same function.

$(2 \times \square) = \triangle$

$(4 \times \square) = (2 \times \triangle)$

$(28 \times \square) = (14 \times \triangle)$

$\square + \square = \triangle$

$(112 \times \square) = (56 \times \triangle)$

$(2000 \times \square) = (1000 \times \triangle)$

$(20{,}006 \times \square) = (10{,}003 \times \triangle)$

A player could select any one of these or one of a thousand others like these, all representing the same function, and we really could not tell which he had picked by examining the table of ordered pairs.

4. This game can be used with all grades successfully by selecting functions of appropriate complexity.

Agenda: Item 5. Addition of Signed Numbers, Using the Number Line

a) $^{-}1 + {}^{-}1 = {}^{-}2$

b) $^{-}3 + {}^{+}3 = 0$

c) $^{-}4 + {}^{+}5 = {}^{+}1$

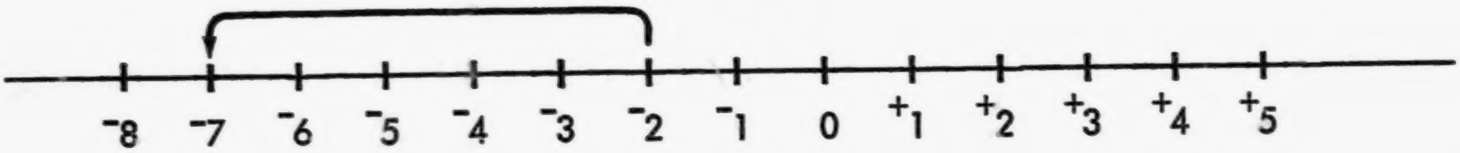
d) $^{+}1 + {}^{+}2 = {}^{+}3$

e) $0 + {}^{-}3 = {}^{-}3$

3

Agenda: Item 6. Postman Stories for Addition

To represent $^{-}2 + {}^{-}5$ on the number line, we would start at $^{-}2$ and move 5 in the negative direction (to the left), landing at $^{-}7$.



Questions for Practice.

a)

$$^{+}5 + {}^{-}3$$

represents a delivery of 2 letters: one contains a check for \$5, and one contains a bill for \$3.

As a result of the delivery, our funds available are up \$2, i.e., we are richer by \$2, so we write

$$^{+}5 + {}^{-}3 = {}^{+}2 .$$

b)

$$^{+}2 + {}^{+}7$$

represents a delivery of 2 letters, one containing a check for \$2, the other containing a check for \$7. As a result, we are \$9 richer, and we write

$$^{+}2 + {}^{+}7 = {}^{+}9 .$$

c)

$$^{-}1 + {}^{-}5$$

represents a delivery of 2 letters, one containing a bill for \$1, the other containing a bill for \$5. Due to the delivery, our funds available are down \$6, i.e., we are \$6 poorer, and we write

$$^{-}1 + {}^{-}5 = {}^{-}6 .$$

d) $^{-}8 + {}^{+}8$

represents a delivery of 2 letters, one containing a bill for $8, and the other a check for $8. As a result of this delivery, we are neither richer nor poorer, so we write

$$^{-}8 + {}^{+}8 = 0.$$

e) $^{+}7 + 0$

represents a delivery of 2 letters, one containing a check for $7, and the other a letter from Aunt Rose, involving no gain or loss of money. As a result, we are $7 richer, and we write

$$^{+}7 + 0 = {}^{+}7 .$$

We should note here that the Postman Stories deal with a different concept than the Pebbles-in-the-Bag or Pet Shop games. In the Pebbles-in-the-Bag game, we used the familiar counting numbers to count stones in and out of the bag. At the end of the counting in and out, we needed a new kind of number. This number had to describe whether there were more or less pebbles in the bag than when we started, and how many more or less. This new kind of number was a signed number.

In the Postman Stories, we took two signed numbers and learned how to combine them.

To sum up, we could say that the Pebbles game is a device to introduce signed numbers, and the Postman game shows us a way to work with them.

SUPPLEMENTARY MODERN MATHEMATICS

For Grades 1 - 9

An In-Service Course for Teachers

SEVENTH SESSION

Agenda:

1. "Postman Stories" vs. "Pebbles-in-the-bag"
2. First Film Excerpt: Postman Stories for Addition and Subtraction
3. Review and Discussion

1. "Postman Stories" vs. "Pebbles-in-the-bag"

WHAT NUMBERS ARISE IN THE COURSE OF *COUNTING*?

Answer: Usually the numbers 1, 2, 3, 4, ... (The three final dots are a mathematician's way of indicating that the pattern goes on "without ever stopping.")

DO SIGNED NUMBERS USUALLY APPEAR IN ORDINARY COUNTING?

Answer: Not the way most people count.

WHERE *DO* SIGNED NUMBERS APPEAR?

Answer: They appear when you have a situation that involves an *arbitrary reference point*, from which you can make measurable deviations in *either of two "opposite" directions.*

Example: On a thermometer, an arbitrary point is labelled "zero." (It *may* be the equilibrium point of ice and water, but need not be.)

The temperature can deviate from this "zero" in either of two "opposite" directions, by measurable amounts.

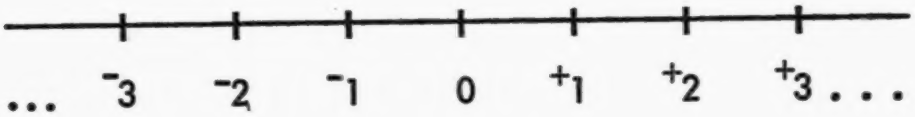
The temperature can be 50 degrees *above* "zero" -- i.e., *warmer.* (Note that some size of *unit* must be selected arbitrarily.)

Alternatively, it can be (say) 10 degrees *below* "zero" -- i.e., *cooler* than zero degrees.

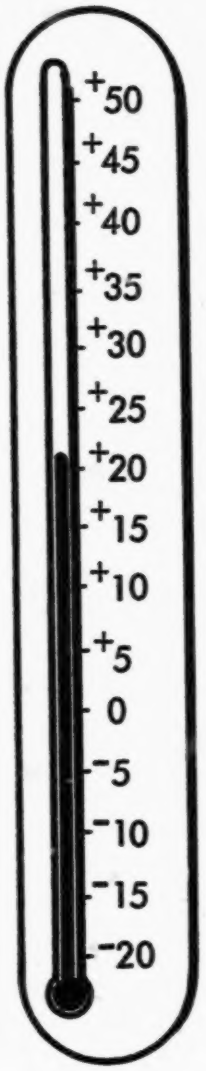
The two directions are "opposite" in the sense that a *rise* of temperature of 10 degrees would just cancel out an earlier *drop* of 10 degrees, and you get back where you started.

(The sophisticated mathematical structure that deals with such situations is called a "one-dimensional linear vector space.")

The usual picture for such a situation is a number line:



Notice that, in fact, a thermometer really uses a piece of the number line, usually vertical rather than horizontal:


⁺50
⁺45
⁺40
⁺35
⁺30
⁺25
⁺20
⁺15
⁺10
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0
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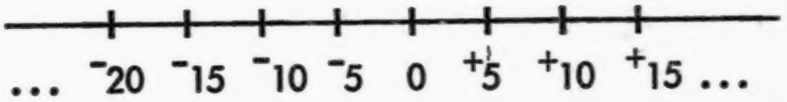
... ⁻20 ⁻15 ⁻10 ⁻5 0 ⁺5 ⁺10 ⁺15 ...

WHAT KINDS OF NUMBERS DOES THE "PEBBLES-IN-THE-BAG" MODEL INVOLVE?

Answer: Actually, this is a rather subtle question, and some rather intricate answers are possible. We shall avoid them, and think in terms of the following answer, which is simple and clear-cut.

When Joe or someone says "Go!" he establishes *an arbitrary reference point*. Thereafter, "zero" will always mean "as many pebbles as there were in the bag when Joe said 'Go!'"

Consequently, the state of the bag -- how "full" or "empty" it is -- will thereafter be described using signed numbers.

$^{+}10$ will mean "10 more pebbles in the bag than when Joe said 'Go!'"

$^{-}3$ will mean "3 less pebbles in the bag now than there were when Joe said 'Go!'" ... and so forth ...

HOWEVER -- when we *put (say) seven* pebbles into the bag, we are *counting*. This, therefore, is "just plain seven" -- i.e., we shall *not* consider it to be a signed number at all.

The same is true when we remove (say) five pebbles from the bag. Again, we are *counting*, so we shall consider this to be "just plain 5," and *not* a signed number at all.

IN OTHER WORDS, if we put 7 pebbles into the bag, we write

$$7$$

and if we then remove 5, we write

$$7 - 5$$

and, since the result of the 2 acts together is that there are now *2 pebbles more in the bag than there were when Joe said "Go!"* we shall write

$$7 - 5 = {}^{+}2.$$

WHAT KIND OF NUMBERS ARE THESE?

$$7 - 5 = {}^{+}2$$

This is just an "ordinary" 7, not a signed number

This is just an "ordinary" 5, and not a signed number

But this is a signed number, namely "positive two."

CONSEQUENTLY, WE SEE THAT THE "PEBBLES-IN-THE-BAG" MODEL PERMITS US TO ADD OR SUBTRACT UNSIGNED NUMBERS, AND TO EXPRESS THE ANSWER AS A SIGNED NUMBER.

Using the "Pebbles-in-the-bag" model, we can do problems like the following:

$$7 - 5 = {}^{+}2$$

$$10 - 15 = {}^{-}5$$

$$3 + 7 = {}^{+}10$$

$$3 + 7 + 2 = {}^{+}12$$

$$3 + 7 - 7 = {}^{+}3$$

$$16 - 20 + 6 = {}^{+}2$$

$9 - 15 + 6 = 0$ (or, if you prefer, either ${}^{+}0$ or ${}^{-}0$)

and so forth.

Remember

$$3 - 5 = {}^{-}2$$

is read

"three minus five equals negative two"

and

$$7 - 3 = {}^{+}4$$

is read

"seven minus three equals positive four."

Similarly,

$$7 + 2 = {}^{+}9$$

would be read

"seven plus two equals positive nine."

The "pebbles-in-the-bag" story corresponding to

$$7 + 2 = {}^{+}9$$

would be:

Joe says "Go!"

You put 7 pebbles into the bag.

Then you put 2 pebbles into the bag.

There are now 9 more pebbles in the bag than there were when Joe said "Go!" ("positive nine" describes the present state of the bag).

ALL THAT WE HAVE SAID THUS FAR RELATES TO THE PEBBLES-IN-THE-BAG MODEL. THE NUMBERS ADDED OR SUBTRACTED ARE UNSIGNED NUMBERS; ONLY THE "ANSWER" IS A SIGNED NUMBER.

NOW WE WISH TO GO FURTHER. WE WANT TO ADD AND SUBTRACT SIGNED NUMBERS.

Instead of problems like

$$5 - 8 = {}^{-}3,$$

for which the "pebbles-in-the-bag" model worked quite well, we now want to move on to more complex problems, such as

$$^{+}7 + {}^{-}3 =$$

$$^{+}8 - {}^{+}2 =$$

$$^{-}3 - {}^{+}6 =$$

$$^{+}1 - {}^{-}3 =$$

and so on.

FOR THESE PROBLEMS, THE PEBBLES-IN-THE-BAG MODEL WILL NOT WORK SATISFACTORILY.

We need a new model, namely, "Postman Stories."

We can represent a check as a positive number, and a bill as a negative number.

"Addition" will correspond to the postman bringing one thing, and also bringing another thing.

Examples:

$^{+}2 + {}^{+}3$	Postman brings a check for \$2, and also brings a check for \$3.
$^{-}5 + {}^{-}3$	Postman brings a bill for \$5, and also brings a bill for \$3.
$^{+}8 + {}^{-}10$	Postman brings a check for \$8, and also brings a bill for \$10.

In carrying these problems further, we can say:

$^{+}1 + {}^{+}6 = {}^{+}7$ This morning the postman brought two letters. One was a check for \$1, and the other was a check for \$6. As a result of his visit this morning, Mrs. Housewife is richer by the amount of \$7.

When a teacher has had sufficient experience, he can use "Postman Stories" as a reliable and foolproof tool for problems involving signed numbers. AT FIRST, HOWEVER, THERE CAN BE PITFALLS. They can be avoided by experience, thought, and (hopefully) group discussions in these In-Service sessions.

Pitfall #1. General confusion over various bills and checks. To avoid this, we introduce time -- i.e., what the Postman brought "this morning," or "sometime last week," or "yesterday," etc. As you read these examples, watch how time is used to help keep some orderliness in the onslaught of bills and checks, and to separate distinct problems. For example, notice the use of "this morning" and "as a result of his visit this morning," in the last example above ($^{+}1 + {}^{+}6 = {}^{+}7$).

Pitfall #2. Confusion in use of subtraction. We introduce subtraction as "error correction." We allow our Postman the whimsical attributes of reading all the mail, delivering much of it to the wrong people, and then later coming around, reclaiming it, and taking it on to its rightful recipient.

Suppose, on Monday morning,[1] the Postman delivers to Mrs. Jones a check for $10.

$$^{+}10$$

and at the same time tells her that the check for $100 which he brought last week was not really for her, and demands it back.

Now, Mrs. Jones had been counting on spending that $100 as part of the payment on a new TV set. In fact, including the $100, Mrs. Jones figured that she had $398.97 toward the cost of the TV; i.e., her "mental bank balance" or available funds were: $398.97.

---

[1] Notice the use of time to help keep things straight ("on Monday morning").

However, after the Postman's visit this Monday morning, Mrs. Jones has an additional $10 check, but has had to return the check for $100. Combining these two transactions, we see that, as a result of the Postman's visit this Monday morning, Mrs. Jones' "available funds" must be reduced by $90.

${}^{+}10 - {}^{+}100 = {}^{-}90$ Available funds: ~~\$398.97~~

\$308.97

The short form of this story would be to say: The Postman brought a check for $10, took away a check for $100, and as a result Mrs. Jones is poorer by the amount of $90.

(Notice that Mrs. Jones is moderately whimsical herself: she never bothers to see whether a check is really for her.)

Can you make up Postman Stories for these problems?

1) ${}^{+}3 + {}^{+}8 = {}^{+}11$

2) ${}^{+}10 + {}^{-}2 = {}^{+}8$

3) ${}^{-}8 + {}^{-}4 = {}^{-}12$

4) ${}^{+}10 - {}^{+}5 = {}^{+}5$

5) ${}^{+}20 - {}^{-}100 = {}^{+}120$ (Be careful on this one!)

6) ${}^{-}10 - {}^{+}2 = {}^{-}12$

7) ${}^{-}10 - {}^{-}6 = {}^{-}4$

8) ${}^{-}10 + {}^{-}6 = {}^{-}16$

(Notice that, in problem #7, the Postman takes away a bill for $6; in problem #8, he brings a bill for $6. If you were Mrs. Jones, which would you prefer?)

YOU MAY NOW WISH TO VIEW THE FIRST FILM EXCERPT.

MADISON PROJECT

Syracuse University • Webster College

SUPPLEMENTARY MODERN MATHEMATICS

For Grades 1-9

REFERENCE BOOK

Seventh Session

The Use of "Mental Imagery"

We have used two "models" to deal with the arithmetic of signed numbers: "postman stories" and "pebbles-in-the-bag." (Actually we have implicitly used also a third model, namely, the *number line*.)

This use of "models" or "mental imagery" is a somewhat distinctive feature of "modern" mathematics teaching. Traditional teaching relied primarily upon explicit rules, such as the "rule for adding numbers of unlike sign," which reads as follows:

> "subtract the smaller from the larger, and use the sign of the larger."

WHAT ADVANTAGES DO YOU SEE FOR EACH APPROACH?

Notice that truly *modern* mathematics teaching *would never use a rule such as the one given above*. Once you have a mental "picture" of how signed numbers work, *you do not need a rule of that sort*.

(Incidentally, the traditional rule was, as a matter of fact, in error, particularly in its use of the words "smaller" and "larger," and in its separation of "numbers" from their "signs." Which is larger, $^{-}21$ or $^{+}2$? What is involved here is a failure to distinguish "a number" from "the absolute value of that same number.")

Postman Stories.

1) ${}^{+}3 + {}^{+}8 = {}^{+}11$

The postman brought a check for \$3, and also brought a check for \$8. As a result of his visit, the housewife is \$11 richer.

2) ${}^{+}10 + {}^{-}2 = {}^{+}8$

The postman brought a check for \$10, and a bill for \$2. As a result, the housewife is \$8 richer.

3) ${}^{-}8 + {}^{-}4 = {}^{-}12$

The postman delivered a bill for \$8 and a bill for \$4. As a result of the delivery, the housewife is \$12 poorer.

4) ${}^{+}10 - {}^{+}5 = {}^{+}5$

On Wednesday morning the postman delivered a check for \$10 and picked up a \$5 check he had left there by mistake last week. As a result of this visit, the housewife is \$5 richer. (Don't forget, last week our housewife would have added that \$5 check into her "mental bank balance" or "funds available," and therefore when she now gives it back, considers herself \$5 poorer for it.)

5) ${}^{+}20 - {}^{-}100 = {}^{+}120$

The postman delivered a check for \$20 and picked up a bill for \$100 which he had left there previously by mistake. Now \$100 need no longer be earmarked for payment of that bill, and as a result, the housewife is richer by \$100 and by \$20; i.e.,

$120 richer.

6) $^{-}10 - {}^{+}2 = {}^{-}12$

The postman delivered a bill for $10 and picked up a $2 check which he had left there previously. As a result of this visit, the housewife is poorer by $12.

7) $^{-}10 - {}^{-}6 = {}^{-}4$

The postman brings a bill for $10 and picks up a $6 bill left by mistake at an earlier date. As a result of this visit, the housewife mentally assigns $10 from her available balance to cover the bill, but now can also add $6 to her balance which formerly was earmarked for the $6 bill. Her available balance goes down $10 and up $6 in quick succession. As a result she is $4 poorer.

8) $^{-}10 + {}^{-}6 = {}^{-}16$

The postman delivers a bill for $10 and a bill for $6. As a result of this visit, the housewife is $16 poorer.

The enclosed tear sheet from the Chicago American should remind us that negative numbers really do appear in print now and then. It's only unfortunate that the symbol " – ," indicating that the temperatures were below zero, was placed so low in relation to the numerals. Maybe a mathematics course for typesetters is in order.

Actually, there is little chance for confusion of the indicated meaning in this instance, but this is not always the case. The horizontal hash-mark symbol, as "traditionally" placed, is used in at least three separate contexts.

1. Subtract: as in 4 subtract 3; i.e., $4 - 3 = 1$
2. Opposite of: as -2 is the opposite of 2; so that $-2 + 2 = 0$
3. Negative: as in -5; i.e., 5 below zero.

This multiple use of the symbol has led to a good deal of confusion for the beginner in the past.

Baffler
PUZZLE
Turn to Page 28

Chicago's AMERI
Always On

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Vol. 64, No. 144 Four-Sections, Section I THURSDAY, DECEMBER 19, 1963 Phone:

COLD? IT MAY HIT -1

Really muffled up!

Chicagoans find varied ways of keeping warm—from face masks

Here's Weather Pi

Christmas Message

Victim's Wife Writes to Slayer

A poignant Christmas message to a fugitive killer has been penned by the bride of a 20-ye ld service station attend was slain in a b Wyatt

Aurora, working 1 to 9 a. m., and then going to classes at barber school.

efore he went to work for time, he asked me to im a few dollars r household ac- e every cent, r future. want

Warning— Beware of Frostbite!

What you don't know won't hurt you, so read no further unless you can face the worst, weatherwise.

The army's "chill factor" chart, which charts the combined effect of temperature and wind on the human body, shows that the combination for Thursday is the e lent of 70- witho r 1-

How Cold Was Your Town?

Elgin	**-17**	**Niles**	**-9**	**Waukegan**	**-7**
Woodstock	**-17**	**Morton Grove**	**-9**	**Cicero**	**-6**
Glenview	**-12**	**Westchester**	**-9**	**Oak Park**	**-5**
Joliet	**-10**	**Barrington**	**-9**	**Skokie**	**-4**
Wheaton	**-10**	**Bedford Park**	**-7**	**Evanston**	**-4**
Norridge	**-10**	**Blue Island**	**-7**	**Gary**	**-4**
Homewood	**-10**	**Berwyn**	**-7**	**Des Plaines**	**-11**

ble stemmea from weak batteries, the Orsis report.

To keep the police department rolling, an order has been issued that squad car engines shall be kept running while the policemen are investigating cases or reporting into their stations. Out-of-service cars at the stations must be warmed up for 15 minutes every hour.

THE WEATHER

Tonight low 8 to 15 below. Tomorrow fair, high 8 to 15 above. North to northwest winds. *(Details on Page 4.)*

Relative Humidity—48%

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ago's AMERICAN
Always On Top Of The News

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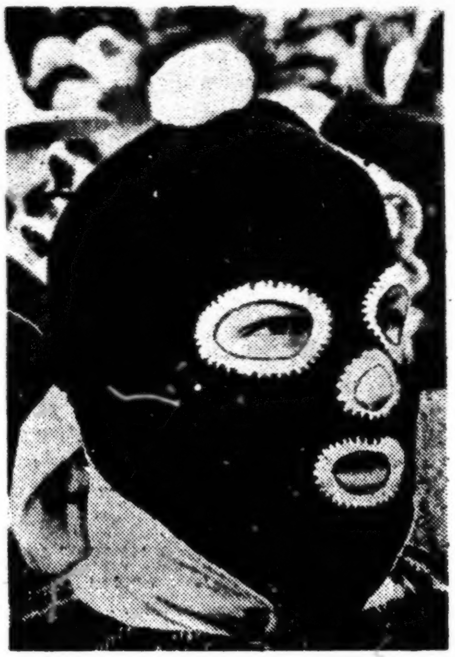
FINAL MARKETS
GREEN STREAK
COMPLETE

ion 1 THURSDAY, DECEMBER 19, 1963 Phone: 222-4321 7 7 CENTS

MAY HIT -15 TONIGHT!

[CHICAGO'S AMERICAN Photos]

s find varied ways of keeping warm—from face masks to Eskimo gloves.



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How Cold Was Your Town?

Elgin	-17	Niles	-9	Waukegan	-7
Woodstock	-17	Morton Grove	-9	Cicero	-6
Glenview	-12	Westchester	-9	Oak Park	-5
Joliet	-10	Barrington	-9	Skokie	-4
Wheaton	-10	Bedford Park	-7	Evanston	-4
Norridge	-10	Blue Island	-7	Gary	-4
Homewood	-10	Berwyn	-7	Des Plaines	-11

ble stemmea from weak batteries, the Orsis report.

To keep the police department rolling, an order has been issued that squad car engines shall be kept running while the policemen are investigating cases or reporting into their stations. Out-of-service cars at the stations must be warmed up for 15 minutes every hour.

THE WEATHER

Tonight low 8 to 15 below. Tomorrow fair, high 8 to 15 above. North to northwest winds. *(Details on Page 4.)*

Relative Humidity—48%

No Relief Seen Before Monday

Chicago's record December cold wave continued into its seventh consecutive day of zero or below Thursday.

And—hold on to your ear muffs—the mercury will plummet to 8 to 15 below Thursday night.

In the suburbs it will
colder. And
sight

'Chaos-at-Large' Doesn't Includ Senate, Say

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SUPPLEMENTARY MODERN MATHEMATICS

For Grades 1 - 9

An In-Service Course for Teachers

EIGHTH SESSION

Agenda:

1. Empirically-determined Functions
2. First Film Excerpt: Graphs and Tables for Weights and Springs
3. DO IT YOURSELF: Weights and Springs

1. Empirically*-determined Functions

In previous lessons we have looked at and approached mathematics in the light of some experiences which children find very exciting. We have in particular talked about signed numbers, variables and open sentences, linear graphs and "rules" of functions. What we will do today will be to use these mathematical experiences in an interesting science experiment.

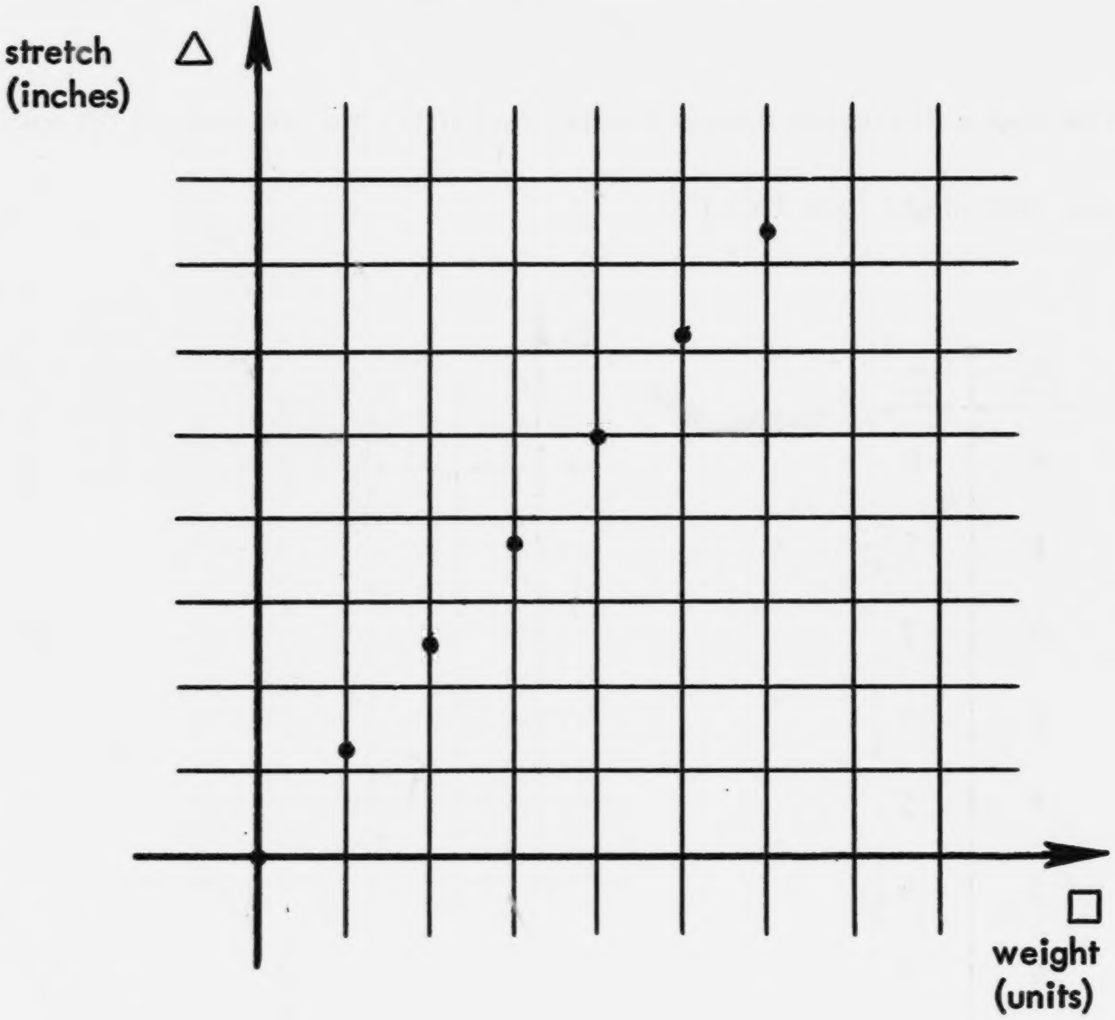
Now not all of physics involves the relationships between numbers, but a large part does, and we shall see in one particular case how we can derive a function from a study of a physical problem.

When we pull down on a hanging spring, it stretches. If we think of a function as a rule, we put a number in the □, use our rule on that number, and get a number in the △. How do we arrive at a function from the spring? We could hang different weights on the spring, then measure the amount the spring stretches. In other words, the number in the □ will be the weight, the number in the △ will be the distance the spring stretches. We can now collect our data, which will look something like this:

□ weight (in units)	△ stretch (inches)
0	0
1	$1\frac{1}{4}$
2	$2\frac{1}{2}$
3	$3\frac{3}{4}$
4	5
5	$6\frac{1}{4}$
6	$7\frac{1}{2}$

* Based on experiment and observation.

Our unit of weight is arbitrary, for we could use 50 gram weights (as in the film), or weigh each washer in our apparatus here in ounces or grams. For the sake of simplicity, we will call the weight of each washer a "unit." That is to say, we shall use the "unit," the weight of one washer, as our unit of measure. Now that we have some pairs of numbers, what can we do with them? We can graph these pairs of numbers and try to write an open sentence for our function. The graph of our data might look like this:



This looks like a linear graph (that is to say, our various points seem to lie along a straight line), and our open sentence will look like this:

$$(\underline{\quad} \times \square) + \underset{\uparrow}{\underline{\quad}} = \triangle.$$

This number must be zero, because this graph goes through (0,0), and we have

$$(\underset{\uparrow}{\underline{\quad}} \times \square) = \triangle.$$

This number tells us how many to go up when we go over one; therefore it must be

$1\frac{1}{4}$ or $\frac{5}{4}$. An open sentence which describes our function then is

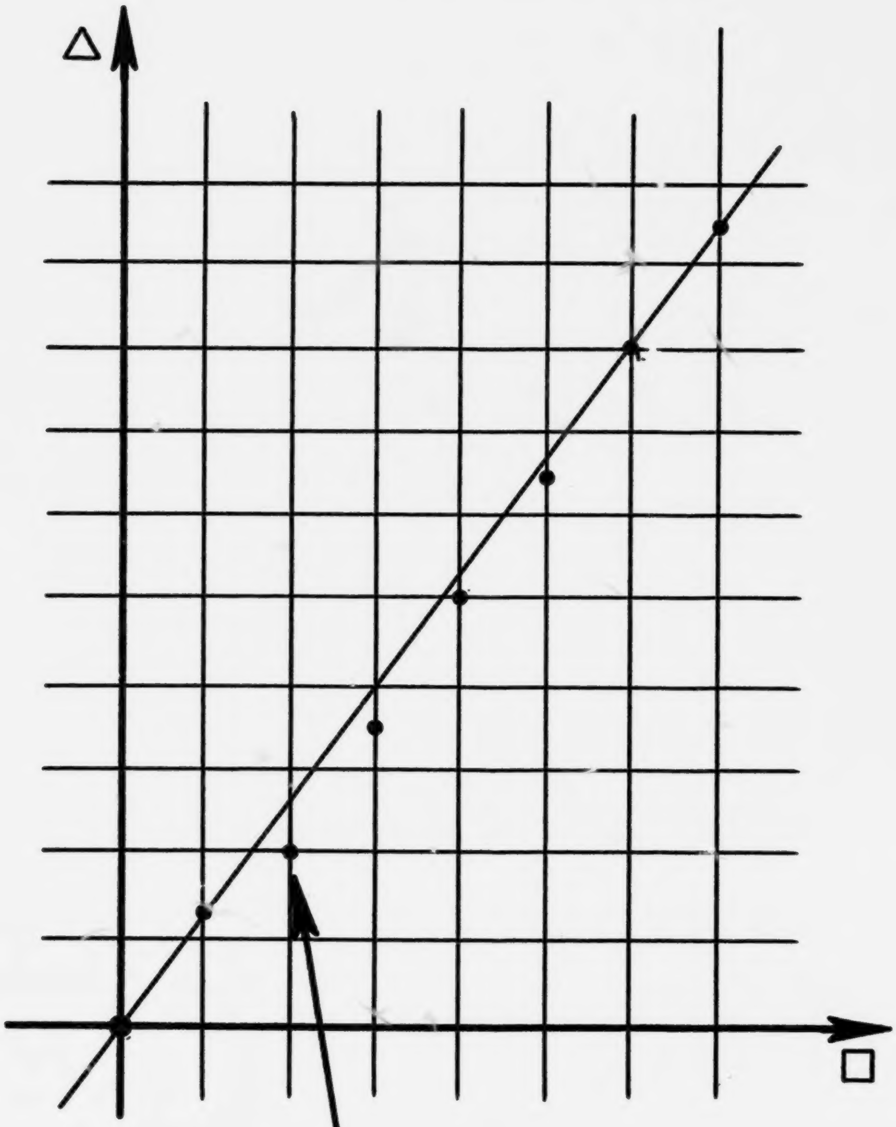
$$\left(\frac{5}{4} \times \square\right) = \triangle.$$

In the film, 50 gram weights are used, and the pattern of the graph is "over 50 to the right, and up $1\frac{7}{8}$ or $\frac{15}{8}$." Consequently, the equation must be

$$\frac{\frac{15}{8}}{50} \times \square = \triangle.$$

The above discussion sounds simple, and it is, but we can run up against the case where our data might look like this:

□	△
0	0
1	$1\frac{1}{4}$
2	2
3	$3\frac{1}{2}$
4	5
5	$6\frac{1}{2}$
6	8
7	$9\frac{1}{2}$



This point doesn't "fit" like the others; it is below the "function line."

The scientist will look at the results of his experiment and ask: "Does my data tell me anything?" "Can I say anything about the relationship between the weight and stretched spring?" -- We have to look at our data and ask the same questions. We might want to do the experiment over again or say to ourselves, "this data, except for one point, is close enough to a straight line; I will accept that one point as experimental error." Some questions we could ask about the experiment are:

What could have caused this point to be off?

How could we get better results?

Questions for further discussion:

Why has this particular experiment been used?

Can we get functions which are not linear?

Can we get functions without measuring anything?

YOU MAY AT THIS POINT WANT TO VIEW THE FILM EXCERPT: Graphs and Tables for Weights and Springs.

3. DO IT YOURSELF: Weights and Springs

Precaution: To get better results for the weights (washers) at hand, the spring probably should be stretched slightly by hand to relieve it of initial tension due to the manufacturing process.

Each set of apparatus (approximately 4 people to a set-up) should consist of:

1 plastic ruler

2 jumbo paper clips

2 tacks

1 spring (.027" music wire, 119 coils, or one similar)

1 rubber band (large)

8 washers (heavy enough to make springs strecth appreciably; we suggest the size shown below)

4 pieces of graph paper

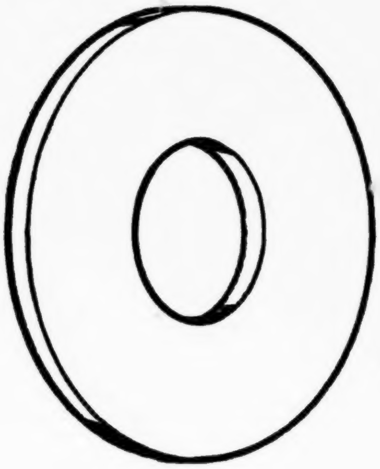
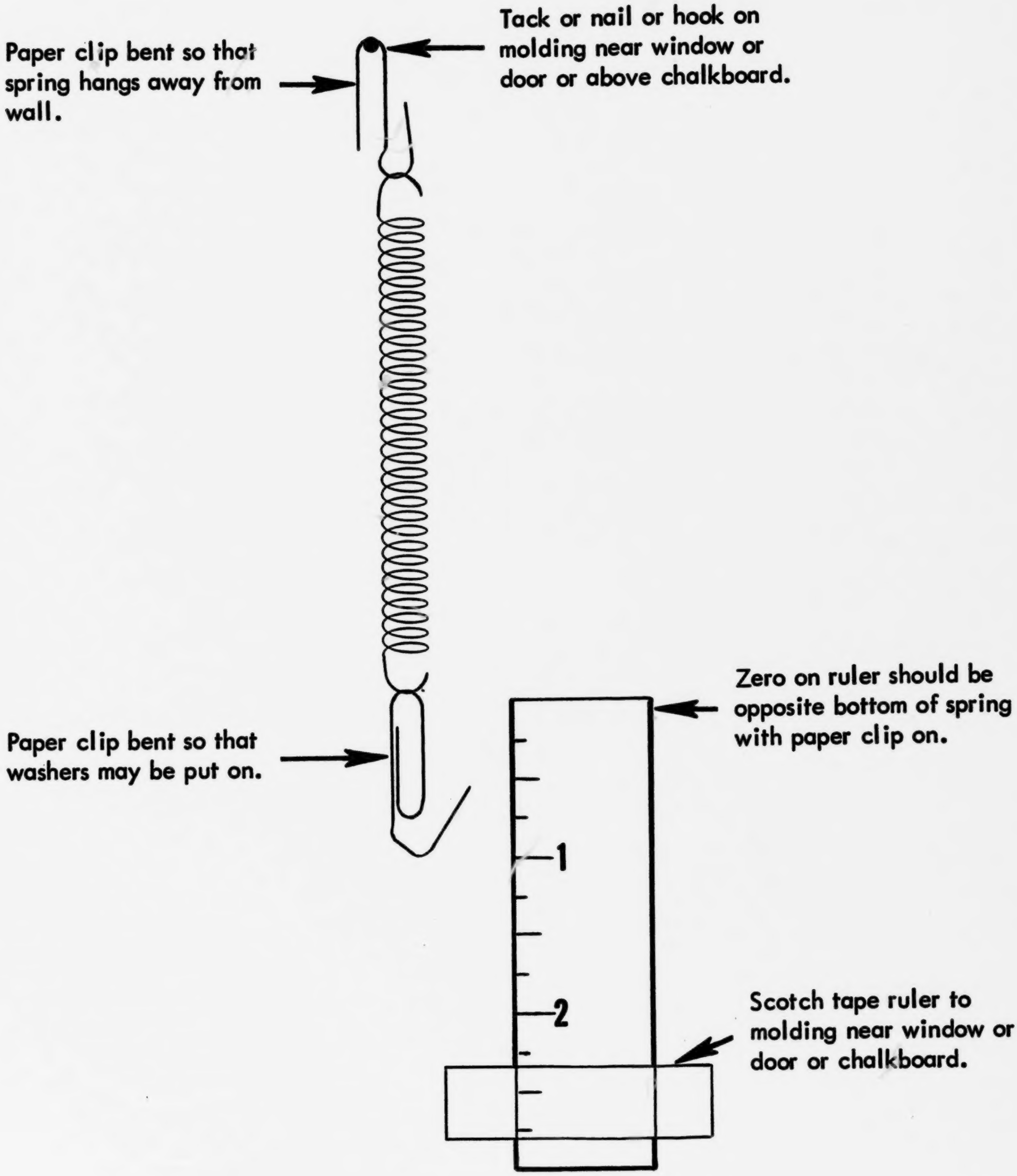


Diagram of Apparatus


Tack or nail or hook on molding near window or door or above chalkboard.
Paper clip bent so that spring hangs away from wall.
Paper clip bent so that washers may be put on.
Zero on ruler should be opposite bottom of spring with paper clip on.
1
2
Scotch tape ruler to molding near window or door or chalkboard.

If time permits you may want to replace the spring with the rubber band, observing what happens when this experiment is performed.

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SUPPLEMENTARY MODERN MATHEMATICS

For Grades 1-9

REFERENCE BOOK

Eighth Session

1. Many people, when they graph the points obtained from the weights-and-spring experiment, find that these points appear to lie along a straight line. By contrast, the points obtained hanging weights on a rubber band instead of a steel spring usually do not appear to lie along a straight line. For this reason, scientists and engineers often refer to "linear elasticity" in the case of the steel spring, and "non-linear elasticity" in the case of the rubber band.

This raises an interesting question: do the points obtained with the spring really lie along a straight line, or do they merely seem to do so? There are many factors to be considered in a detailed analysis of this problem, including whether all of the weights were identical, whether the paper-clip or other clamping arrangement that held the spring may have shifted or moved, whether the ruler used was accurate, whether the person reading the ruler did in fact read it correctly, as well as how the steel spring itself may have responded to the stresses. Would the spring be affected by changes in temperature?

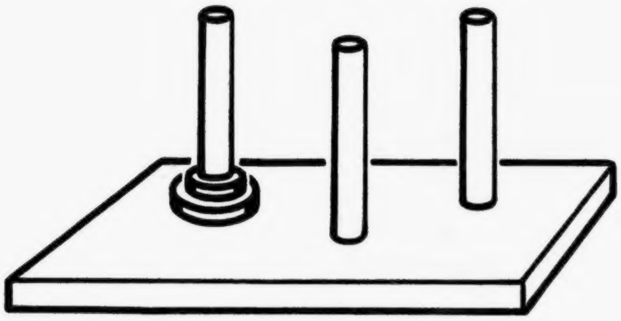
2. If we assume that it is reasonable to say that the points obtained from the steel spring do appear to lie along a straight line, then this experiment gives us an example of what is called a linear function. This leads easily to the concept of direct variation [as the weight is doubled, the amount the spring stretches is doubled -- if we start at (0,0)]. There are other linear functions we could have used instead, such as centigrade vs. fahrenheit temperature readings obtained while heating a liquid, circumference vs. diameter for various circles (to get π), and weight vs. volume for various samples of a metal (density).

There are also many non-linear functions which can be obtained empirically in mathematics and science, such as the distance vs. time for a freely falling object, the area of a square vs. its side, the sine of an angle vs. the angle.

3. We have in previous lessons been getting functions without measuring anything. We could also get empirically-determined functions by counting. For example, the Tower of Hanoi puzzle provides an effective way to get a function whose equation is

$$2^{\square} - 1 = \triangle$$

if the $\square$-number is the number of discs and the $\triangle$-number is the number of moves needed to bring all discs to another peg. The discs must be moved one at a time and a larger disc must never be placed on top of a smaller one.



4. In terms of classroom organization, it might be worthwhile to have the youngsters work in smaller groups on two or three experiments, for maximum student involvement.

SUPPLEMENTARY MODERN MATHEMATICS

For Grades 1 - 9

An In-Service Course for Teachers

NINTH SESSION

Agenda:

1. Postman Stories for Addition
2. First Film Excerpt: Postman Stories for Addition
3. Postman Stories for Multiplication
4. Second Film Excerpt: Postman Stories for Multiplication
5. Postman Stories for Subtraction
6. Third Film Excerpt: Postman Stories for Addition, Subtraction, and Multiplication
7. Discussion
8. "Taking Away"
9. Fourth Film Excerpt: Postman Stories that Involve "Taking Away"
10. Discussion

1. Postman Stories for Addition

We have seen, in previous sessions, that it will be valuable to have some mental symbols that correspond to such problems as

$$^{+}5 + {}^{+}3$$

$$^{+}5 + {}^{-}2$$

$$^{-}7 + {}^{-}2$$

$$^{+}7 - {}^{+}5$$

$$^{+}6 - {}^{-}3$$

$$^{-}4 - {}^{-}8$$

and so on.

We have also begun to see (in sessions 6 and 7) how to use a postman, delivering and picking up mail, and a housewife who receives and returns mail, as a model for arithmetical porblems of this type.

In the present session we explore this idea more fully.

In the first place, we agree to let positive numbers correspond to checks, and to let negative numbers correspond to bills. Thus

$^{+}7$ ("positive seven")

would correspond to a check for $7, and

$^{-}3$ ("negative three")

would correspond to a bill for $3.

In the film excerpt which follows, you will see three arithmetical problems worked out, using the "postman" model.

The first arithmetical problem is

$$^{+}7 + {}^{+}3.$$

At the outset, a "housewife" believes herself to have \$1 available as spending money.

housewife's record of
available spending money

\$1

At this point, the "postman" arrives, bringing a check for \$7 and a check for \$3:

$$^{+}7 + {}^{+}3.$$

As a result of this visit by the postman, will the housewife be richer or poorer? By how much?

The answer, of course, is that she will be richer by \$10:

$$^{+}7 + {}^{+}3 = {}^{+}10,$$

(where the $^{+}10$ indicates "richer by \$10"), and so the housewife changes her estimate of her available spending money:

record of available money

~~\$1~~
\$11

Two other arithmetical problems app[illegible] this section of the film, and are handled similarly:

$$^{+}7 + {}^{-}5 = {}^{+}2$$

record of available money

~~\$64~~
\$66

YOU MAY NOW WISH TO VIEW THE FIRST FILM EXCERPT.

3. Postman Stories for Multiplication

We know that 2 inches is multiplied by 3 inches in finding the area of certain rectangles, and the answer is 6 square inches.

Keeping this in mind, we can turn to problems in the multiplication of signed numbers, such as

$$^{+}2 \times {}^{+}5.$$

Now, if we were to regard the $^{+}2$ as corresponding to \$2, and the $^{+}5$ as corresponding to \$5, then evidently

$$^{+}2 \times {}^{+}5$$

would correspond to something like "ten square dollars." This, of course, is nonsense. We shall, consequently, have to seek a different interpretation for

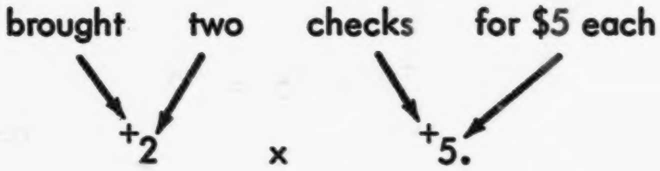
$$^{+}2 \times {}^{+}5.$$

Fortunately, a suitable interpretation lies immediately at hand: "we multiply by 2" when we "do something twice." Let us agree that

$$^{+}2 \times {}^{+}5$$

will mean: "the postman came this morning and brought two checks for \$5 each."

Notice that the correspondence then works as follows:



In the film excerpt which follows, two problems in the arithmetic of signed numbers are represented by suitable "postman stories." In the first problem, the housewife starts with an estimated \$97 of available money:

housewife's record
of available money

\$97

The postman then brings 2 checks for \$3 each:

$$^{+}2 \times {}^{+}3.$$

As a result, the housewife is \$6 richer

$$^{+}2 \times {}^{+}3 = {}^{+}6,$$

and she changes her account book accordingly:

housewife's record
of available money

~~\$97~~

\$103

In the second example, the housewife begins with \$50:

housewife's record of
available spending money:

\$50

The arithmetical problem to be solved is

$$^{+}7 + {}^{-}10.$$

Since this is an addition problem, we return to our earlier model, and represent $^{+}7$ as a check for \$7, and $^{-}10$ as a bill for \$10. The "postman story" then says: "The postman came this morning, and brought a check for \$7 and a bill for \$10. As a result of his visit this morning, the housewife is poorer by \$3." Consequently, we write:

$$^{+}7 + {}^{-}10 = {}^{-}3,$$

and the housewife changes her records accordingly:

**housewife's estimate of
available spending money**

~~$50~~

$47

YOU MAY NOW WISH TO VIEW THE SECOND FILM EXCERPT.

5. Postman Stories for Subtraction

Arithmetical problems such as

$$^{+}7 - {}^{+}5$$

we shall interpret by saying: "The postman came this morning and brought a check for $7. He also took away a check for $5. As a result of his visit this morning, is the housewife richer or poorer?"

In the film excerpt which follows, three arithmetical problems are solved. In the first problem, the housewife starts with $20 of apparently available spending money:

housewife's record of apparently
available spending money:

$20

At this point, the postman arrives, bringing a check for $7, but taking away a check for $5 that had previously been delivered in error (and which the housewife had included in her estimate of "apparently available spending money"). *As a result of the postman's visit this morning*, the housewife must revise her estimate of "available spending money" upward by the amount of $2.

record of apparently
available spending money:

~~$20~~

$$^{+}7 - {}^{+}5 = {}^{+}2$$

$22

In the second problem, we again encounter multiplication. Written in brief schematic form, the problem might be summarized as:

estimated available money

~~$10~~

$^{+}5 \times {}^{-}3 = {}^{-}15$

$^{-}\$5$

Can you figure this out, before viewing the film?

Finally, in the third problem we again have a "subtraction" or "taking away" problem:

$^{+}20 - {}^{-}4.$

The housewife begins with $54 of apparently available spending money:

housewife's record of apparently
available spending money

$54.

This morning, however, the postman arrives, bringing a check for $20, and taking away with him a bill for $4 that he had previously delivered in error (in this Kafka-like story, it is of course the postman, not the housewife, who discovers and announces these "errors"). As a result of the postman's visit, including his announcement that the bill for $4 (which the housewife had included in computing her record of "apparently available money") was really for someone else, the housewife should now revise her estimate of available funds upward by the amount of $24:

housewife's record of apparently
available spending money:

~~$54~~

$$^{+}20 - {}^{-}4 = {}^{+}24$$

$78.

YOU MAY NOW WISH TO VIEW THE THIRD FILM EXCERPT.

7. Discussion

You may wish to try "teaching" some "postman story" problems as you would to your own students, and then having the group discuss them.

8. "Taking Away"

Several remarks need to be made about the use of "postman and housewife" models for the arithmetic of signed numbers.

1. There are certain spots in using "postman stories" where the teacher must be extremely careful, or else the "story" won't work out correctly. However, it *is* possible to use postman stories so that they work correctly for every type of problem in adding, subtracting, and multiplying signed numbers.

2. Notice that the stories for

$$^{+}8 + {}^{-}2$$

and

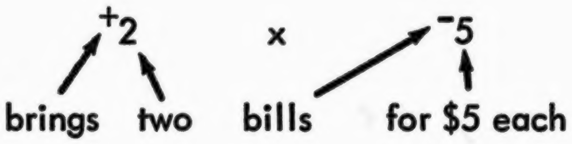
$$^{+}8 - {}^{+}2$$

are *different stories*! (Can you get each one right?)

3. Remember that we use a different procedure with multiplication stories than we do with addition and subtraction stories. In *multiplication* stories, the number on the left *does not represent a bill or a check*, but rather *how often* the postman delivers or reclaims bills or checks:

$^{+}2 \times {}^{+}5$	postman brings 2 checks for $5 each
$^{-}2 \times {}^{+}5$	postman takes away 2 checks for $5 each
$^{+}2 \times {}^{-}5$	postman brings two bills for $5 each
$^{-}2 \times {}^{-}5$	postman takes away two bills for $5 each

The postman stories for a product read continuously from left to right



4. The level of sophistication that you use with your own class must, of course, be adjusted to the appropriate level for your particular students.

A fully consistent "postman" model, that never involves double counting, can be made by using a somewhat surrealistic mail service that reads like a novel by Franz Kafka. In this village (or city or whatever), the postman reads all mail before delivering it. After reading it, he delivers it to whatever address he chooses -- perhaps the correct address, and perhaps not. Whenever he mis-delivers mail, he subsequently corrects the error by appearing at the housewife's door and saying:

"Gosh, Mrs. Wilson, I hope you haven't been planning on spending that $200 check I brought you last week. It wasn't really for you, it was really for a Mrs. Jones on the other side of town. If you'll give it back to me, I'll take it to her now."

... or whatever else may be appropriate, such as (perhaps):

"Golly, Mrs. McKnight, I hope you haven't been worrying about that bill for $50 that I brought you last week. That was really for Mrs. Parsons. If you'll give it back to me now, I'll run over to her place with it this morning."

On their part, the housewives behave in an equally Kafka-esque fashion: they never read the addresses on their mail, but merely notice whether the item is a bill or a check, and for how much (but not for whom). They keep their accounts meticulously up to date, including every bill and check that is delivered to them in their calculations. They never actually cash checks or actually pay bills, but instead keep an updated record of "apparently available spending money." This record must, of course, be changed whenever the postman comes to reclaim a bill or a check.

In the following film excerpt, two problems are solved, which we represent schematically as:

Problem 1

housewife's apparently
available money:

~~$88~~

$$^{-}2 \times {}^{+}3 = {}^{-}6$$

$82

Problem II

housewife's record of
apparently available money:

~~$48~~

$$^{-}3 \times {}^{-}5 = {}^{+}15$$

$63

Can you figure out each problem, with its corresponding postman story, before viewing the film?

YOU MAY NOW WISH TO VIEW FILM EXCERPT NUMBER FOUR.

10. Discussion

We leave this discussion up to you.

SUPPLEMENTARY MODERN MATHEMATICS

For Grades 1 - 9

An In-Service Course for Teachers

TENTH SESSION

Agenda:

1. "Kye's Arithmetic": What does it mean to be "modern"?

2. First Film Excerpt: Kye's Method of Subtracting.

3. Non-standard numerals, and non-standard algorithms. Part 1.

1. "Kye's Arithmetic": What does it mean to be "modern"?

Any teacher working through the present course -- indeed, given the recent hue and cry, it is safe to say any elementary teacher whatever -- must ask himself what all of this "modern mathematics curricula" activity is all about. In fact, without seriously confronting this question (which is more than a one-week task), one can make little constructive use of the material presented here.

There are many alternative answers to this question, and it is not evasion to claim that several different answers deserve consideration.

In today's lesson, however, we shall attempt an answer by way of a single example:

The scene is a third grade class in Weston, Connecticut. The teacher is explaining the subtraction problem

$$\begin{array}{r} 64 \\ \underline{-\ 28} \end{array}$$

Teacher: You can't subtract 8 from 4, and so you transfer 10 from the 60, ...

Kye -- a third-grade boy -- interrupting: "Oh, yes you can! Four minus eight equals negative four

$$\begin{array}{r} 64 \\ \underline{-\ 28} \\ {}^{-}4 \end{array}$$

and 60 minus 20 equals 40

$$\begin{array}{r} 64 \\ \underline{-\ 28} \\ {}^{-}4 \\ \underline{40} \end{array}$$

and so the answer is 36

$$\begin{array}{r} 64 \\ -\,28 \\ \hline {}^{-}4 \\ 40 \\ \hline 36.\text{"} \end{array}$$

Now, that teacher was *modern*! Why? After all, it was Kye, and not the teacher, who invented this delightful algorithm for subtracting.

In our view, that teacher was "modern" because:

1. She had previously given Kye the necessary background in signed numbers so that he was *able* to invent his original algorithm.

2. She had previously given Kye a creative attitude toward mathematics, or else he would not have tried to show the teacher a new way to subtract.

3. Especially this: when Kye claimed that he had a "better" way to do the problem, THE TEACHER LISTENED TO WHAT HE HAD TO SAY, AND WAS ABLE TO EVALUATE IT CORRECTLY.

The "old-fashioned" or "traditional" teacher would have told Kye "No, that's not the way we do it!" Where would that leave Kye?

4. Perhaps there is an even stronger reason for saying that this teacher was "modern": when Kye confronted her with an enexpected new way to solve an old problem, *the teacher was excited and pleased by Kye's contribution*.

An "old-fashioned" teacher might well have resented it.

YOU MAY NOW WISH TO VIEW THE FIRST FILM EXCERPT.

3. Non-standard numerals, and non-standard algorithms. Part 1.

We showed "Kye's method" to some 5th grade children in Ladue, Missouri, and they modified the method as follows:

$$\begin{array}{r} 64 \\ -\ 28 \\ \hline 4\bar{4} \end{array} \; ,$$

where a negative sign written above a digit indicates that that particular digit is negative. That is,

$$4\bar{4} \text{ means } 40 + {}^{-}4 = 40 - 4 = 36.$$

This, after all, is merely our familiar place-value notation in a new role, for

$$36 \text{ means } 30 + 6.$$

Problems:

1. Can you "find the sum" by two different methods?

$$\begin{array}{r} 3\bar{5} \\ +\ 2\bar{1} \\ \hline \end{array}$$

(addition problem,
involving
negative digits)

2. Can you "subtract" by two different methods?

$$\begin{array}{r} 7\bar{9} \\ -\ 4\bar{2} \\ \hline \end{array}$$

(subtraction problem,
involving
negative digits)

3. Can you "find the sum" by two different methods:

a.

$$\begin{array}{r} 5\bar{3} \\ +\ 24 \\ \hline \end{array}$$

(note that only the $\bar{3}$ is a negative digit)

b.

$$\begin{array}{r} 106\bar{6} \\ +\ 17\bar{3}2 \\ \hline \end{array}$$

(addition problem: two of the digits are negative digits; all others are positive digits)

(This topic will be continued in Session Eleven.)

SUPPLEMENTARY MODERN MATHEMATICS

For grades 1 - 9

An In-Service Course for Teachers

ELEVENTH SESSION

Non-Standard Numerals and Non-Standard Algorithms

Agenda:

1. Answers to Questions from Tenth Session
   (Note on the meaning of "equality.")

2. Non-Standard Numerals in General
   - i. Non-Decimal Numerals
   - ii. Numerals with Negative Digits
   - iii. Roman Numerals
   - iv. "Volkswagen" Numerals

3. Non-Standard Algorithms
   - i. Kye's Algorithm for Subtraction
   - ii. Algorithm with Negative Digits
   - iii. The "How Much Must I Add" Algorithm for Subtraction
   - iv. The "Volkswagen" Algorithm

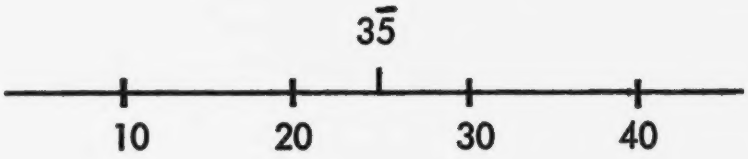
1. Answers to Questions from Tenth Session

Question #1. Find the sum by two different methods:

$$\begin{array}{r} 3\bar{5} \\ +\ 2\bar{1} \\ \hline \end{array}$$

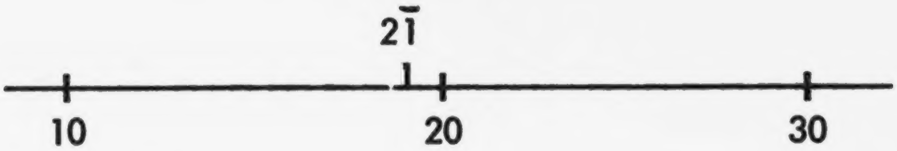
(an addition problem, involving negative digits)

Answer: 1st method. We recall that $3\bar{5}$ means $30 - 5$, or $30 + {}^{-}5$,



which, of course, is also $20 + 5$, or 25.

Similarly, $2\bar{1}$ means $20 + {}^{-}1$, or $20 - 1$



which is also $10 + 9$, or 19.

Consequently, we may, if we wish, handle the addition problem by converting the non-standard numerals $3\bar{5}$ and $2\bar{1}$ into their standard numeral equivalents: 25 and 19. The problem

$$\begin{array}{r} 3\bar{5} \\ +\ 2\bar{1} \\ \hline \end{array}$$

becomes, then,

$$\begin{array}{r} 25 \\ +\ 19 \\ \hline \end{array}$$

and, in the usual way, we "find the sum" to be

$$\begin{array}{r} 25 \\ +\ 19 \\ \hline 44. \end{array}$$

Note: Modern mathematicians use the symbol "=" to mean that whatever name is on the left of the equals sign names the same thing as the name on the right of the equals sign does. For example,

$$\text{IV} = 4$$

means that " IV " names the same number that "4" does. Similarly,

$$3\bar{5} = 25$$

says that $3\bar{5}$ names the same number that 25 does.

Again,

George Washington = the 1st President of the United States

says that "George Washington" names the same person that the phrase "the 1st President of the United States" names. Also, 1 + 1 is the name of a number, and, in fact,

$$1 + 1 = 2.$$

What does "1 + 1 = 2" mean, in terms of names for numbers?

Returning to the addition problem

$$\begin{array}{r} 3\bar{5} \\ \underline{+\ 2\bar{1}} \end{array},$$

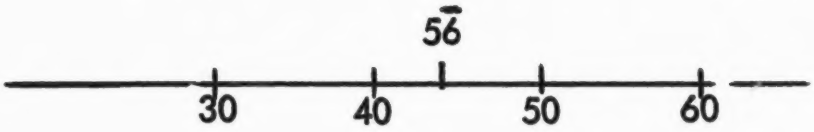
we could instead use a second line of attack, by saying

$$\begin{array}{r} 3\bar{5} \\ \underline{+\ 2\bar{1}} \\ \bar{6} \end{array} \qquad\qquad {}^-5 + {}^-1 = {}^-6$$

and 30 + 20 = 50,

$$\begin{array}{r} 3\bar{5} \\ +\ 2\bar{1} \\ \hline 5\bar{6}\ , \end{array}$$

so the answer is $5\bar{6}$. This, of course, is a non-standard numeral, but it is a perfectly good answer for all that. Indeed, $5\bar{6}$ names the same number that 44 does,



so we could write

$$5\bar{6} = 44\ ,$$

but we need not do this, if we prefer to leave the answer as $5\bar{6}$.

Question #2. Subtract by two different methods:

$$\begin{array}{r} 7\bar{9} \\ -\ 4\bar{2} \\ \hline \end{array}\ .$$

Answer: First Method -- Using non-standard numerals.

$$^-9 - {}^-2 = {}^-7\ ,$$

so we write

$$\begin{array}{r} 7\bar{9} \\ -\ 4\bar{2} \\ \hline \bar{7}\ , \end{array}$$

and $70 - 40 = 30$, so we write

$$\begin{array}{r} 7\bar{9} \\ -\ 4\bar{2} \\ \hline 3\bar{7}. \end{array}$$

Evidently, $3\bar{7}$ is a name (non-standard, to be sure) for the "answer."

(Incidentally, if you prefer, arrange the work on the paper as

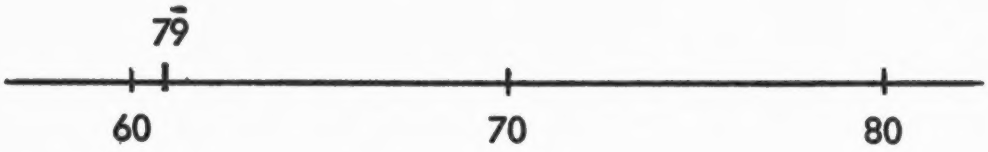
$$\begin{array}{r} 7\bar{9} \\ -\ 4\bar{2} \\ \hline {}^{-}7 \\ 30 \\ \hline 3\bar{7}\ . \end{array}$$

Still other arrangements are possible.)

Second Method -- Conversion to standard numerals:

$7\bar{9}$ means $70 - 9$ or $70 + {}^{-}9$,

and as we see from the number line,



$7\bar{9}$ names the same number that $60 + 1$, or 61, does:

$$7\bar{9} = 61.$$

Similarly,

$$4\bar{2} = 38.$$

Consequently, the problem

$$\begin{array}{r} 7\bar{9} \\ -\ 4\bar{2} \\ \hline \end{array}$$

can be re-written as

$$\begin{array}{r} 61 \\ -\ 38 \\ \hline \end{array},$$

and can be handled in the usual way, using some standard algorithm:

$$\begin{array}{r} 61 \\ -\ 38 \\ \hline 23 \end{array}.$$

Comparison of the two methods: did each method lead to the same answer? Evidently, since $3\bar{7}$ clearly names the same number that 23 names.

i.e., $3\bar{7} = 23$.

For a discussion of the remaining problems from the previous session, please consult the Yellow Reference Book for Session Eleven.

2. Non-Standard Numerals in General

For Americans in 1964, our "standard" numerals are, of course, place-value numerals to base 10, i. e., based upon a grouping by tens. This is often demonstrated by taking, for example, a large pile of tongue-depressors or other suitable objects, and bundling them into bundles of 10 objects each. If, in the course of doing so, we accumulate 10 bundles, then we collect them into one "super-bundle" consisting of exactly 10 bundles. The outcome might be:

"super-bundles"	bundles	loose objects
2	7	3

Such a result can be represented by dropping the names "super-bundles," "bundles," and "loose objects," and writing simply

273,

which would mean precisely the same thing.

Question #1: Suppose Jean had 5 "super-bundles," 3 "bundles," and 9 loose tongue depressors. Suppose Jacqueline had 1 "super-bundle," 4 "bundles," and 7 loose tongue depressors. How many "super-bundles," "bundles," and loose tongue depressors would result if Jean and Jacqueline combined their hoards?

Question #2: Explain the addition

$$\begin{array}{r} 192 \\ +\ 235 \\ \hline 427 \end{array}$$

in terms of the language of "tongue depressors," "bundles," and "super-bundles."

Decimal numerals can also be explained in terms of "expanded notation."

$$1964 = 1000 + 900 + 60 + 4$$

or even

$$1964 = (1 \times 1000) + (9 \times 100) + (6 \times 10) + (4 \times 1)$$

or even

$$1964 = (1 \times 10^3) + (9 \times 10^2) + (6 \times 10^1) + (4 \times 10^0).$$

They can be developed with children using Cuisenaire rods,[1] and even more fully by using the "MAB" or "Multi-base Arithmetical Blocks" developed by Professor Z. B. Dienes for work in the schools in Leicestershire, England.[2] Anyone interested is referred to the references listed in the footnotes. For a discussion of the extended notation used above, consult

Brumfiel, Eicholz, and Shanks, Algebra 1,
Addison-Wesley Publishing Co., Inc., 1961, pp. 4,5.

i. Non-Decimal Numerals. If we replace our bundling into bundles of 10 by, say, the process of making bundles of 5 tongue depressors, "super-bundles" of 5 bundles, etc., we obtain "numerals to base 5." Again, if we make bundles of 2, and super bundles of 2 bundles, etc., we obtain "numerals to base 2" or "binary" numerals. (Our usual numerals to base 10 are also referred to as "decimal numerals.")

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[1] "Cuisenaire rods" may be obtained from:

Cuisenaire Company of America, Inc.
9 Elm Avenue
Mount Vernon, New York

[2] "MAB" blocks are available from:

Learning Materials, Inc.
100 East Ohio Street
Chicago, Illinois 60611

For a discussion of non-decimal numerals, consult

Brumfiel, Eicholz, and Shanks, Algebra 1,
Addison-Wesley Publishing Co., Inc., 1961, pp. 6-9.

Question #3. Using bundles of 3, "super-bundles" of 3 bundles, etc., discuss the following problem.

Doris has 1 "super-bundle," 2 bundles, and 1 loose tongue depressor. Roy has 2 bundles, and 1 loose tongue depressor. How many bundles, "super-bundles," and loose tongue depressors would Doris and Roy have if they combined their possessions?

Non-decimal numerals have received a great deal of attention in recent years, and many excellent references are now available. Particular emphasis has been placed upon binary numerals, and upon numerals to base 12.

ii. Numerals with Negative Digits. These, of course, are merely the numerals discussed at the beginning of the Eleventh Session, and also in the Tenth Session.

iii. Roman Numerals. One of the best-known, and least efficient, systems of numerals is, of course, the system of "Roman Numerals": I , II , III , IV , V , VI , VII , VIII , IX , X , XI , ...

Notice that the decimal numeral 7 names the same number as the non-standard numeral $1\bar{3}$, or as the Roman numeral VII . Consequently, we could write

$$7 = 1\bar{3} = \text{VII}.$$

There are, of course, many other names for this same number. For example

$$6 + 1$$

is a name for this number, and so is 9 − 2, etc. This, of course, is precisely what the modern mathematician means when he writes

$$6 + 1 = 7$$

or

$$9 - 2 = 7 .$$

(An excellent reference on this point is: L. Clark Lay, Times of the Times, The Arithmetic Teacher, vol. 10, no. 6 (October, 1963), pp. 334 - 338.

Question #4. In the box at the right are written various names for various numbers. How many different names are represented? How many different numbers are named? (Do not count parts of names, but only "complete" names).

20_3
$\frac{12}{2}$
(9 − 3)
2
II
5 + 1
1 + 1
VI
6
$1\bar{4}$
2

iv. "Volkswagen Numerals." Various jokes have been made about parking 2 Volkswagens at one parking meter. We can adopt this idea to the construction of a new kind of numeral.

In our usual decimal numerals,

1964
↑
this is the "one's" column

1964
↑
this is the "ten's" column

1964
↑
this is the "hundred's" column

1964
↑
this is the "thousand's" column.

Consequently, 1964 means: "1 thousand, 9 hundreds, 6 tens, and 4 ones."

By distinguishing large digits from small digits, we can write 2 digit numbers in a single column. For example

$$18_{16}4$$

will be regarded as meaning:

$$18_{16}\underset{\uparrow}{4}$$

this is the one's column

$$18_{\underset{\uparrow}{16}}4$$

this is the ten's column

$$1\underset{\uparrow}{8}_{16}4$$

this is the hundred's column

$$\underset{\uparrow}{1}8_{16}4$$

this is the thousand's column

Consequently,

$$18_{16}4$$

means: "one thousand, 8 hundreds, 16 tens, and 4 ones."

Question #5. Which numeral means the larger number, the "usual decimal numeral"

$$1964$$

or the "Volkswagen numeral"

$$18_{16}4_{?}$$

Question #6. Can you perform the following subtraction using "Volkswagen numerals," and express the answer as a standard decimal numeral?

$$\begin{array}{r} 12_{12}7 \\ -\,1196 \\ \hline \end{array}$$

3. Non-Standard Algorithms

Various standard algorithms for adding, subtracting, multiplying, and dividing are well-known, and will not be discussed here.

What we shall discuss are three "non-standard" algorithms made up by elementary school children, plus one "non-standard" algorithm made up by a professor of mathematics. Two of these algorithms have been considered earlier, in the present Session and in Session Ten.

The point is, of course, not to argue which algorithm is best. The point is that mathematics is mainly a creative subject, and that this creativity extends even as far as to allowing students to invent their own original algorithms for adding, subtracting, etc. In more advanced study of mathematics, students will be expected to find new kinds of problems, never previously solved, and to devise new methods for their solution. It is probably desirable to begin this attitude early.

We shall not discuss this at length here, leaving it up to teachers to consider reasons why students should (or should not) be encouraged to devise their own original algorithms, probably as a prelude to sharing, learning algorithms made up by other students, and ultimately learning "standard" algorithms, many of which are centuries old.

What we shall do here is to focus on mathematical aspects of a few "new" algorithms,and open the door to the creation of still other "new" algorithms by those who may wish to pursue this notion.

i . Kye's Algorithm for Subtraction. This algorithm, which we have studied at some length above, was invented by a 3rd grade boy (named Kye) in Weston, Connecticut.

For the problem

$$\begin{array}{r} 64 \\ \underline{-\ 28} \end{array}\ ,$$

Kye proceeded as follows:

$$4 - 8 = {}^-4$$

so we write

$$\begin{array}{r} 64 \\ \underline{-\ 28} \\ {}^-4 \end{array}$$

$$60 - 20 = 40$$

so we write

$$\begin{array}{r} 64 \\ \underline{-\ 28} \\ {}^-4 \\ 40 \end{array}$$

40 and $^-4$ combine to give 36

so we write

$$\begin{array}{r} 64 \\ \underline{-\ 28} \\ {}^-4 \\ \underline{40} \\ 36\ . \end{array}$$

Question #7. Use Kye's algorithm to subtract:

$$\begin{array}{r} 1964 \\ -\ \ 799 \\ \hline \end{array}\ .$$

ii. Algorithm with Negative Digits. This algorithm was made up by 5th graders in Ladue, Missouri. It is an outgrowth of Kye's algorithm.

For the subtraction problem

$$\begin{array}{r} 64 \\ -\ 28 \\ \hline \end{array}\ ,$$

the Ladue algorithm proceeds like this:

$4 - 8 = {}^{-}4$

so we write

$$\begin{array}{r} 64 \\ -\ 28 \\ \hline \bar{4} \end{array}$$

$60 - 20 = 40$

so we write

$$\begin{array}{r} 64 \\ -\ 28 \\ \hline 4\bar{4} \end{array}$$

and $4\bar{4}$ names the answer (using, of course, numerals with negative digits.)

Question #8: Add, by 2 different methods (or, if you can, by 3 methods):

$$\begin{array}{r} 19\bar{6}\bar{5} \\ +\ 73\bar{4}5 \\ \hline \end{array}$$

addition, using
negative digits

iii. The "How Much Must I Add" Algorithm. This algorithm was an original invention of some 6th grade students in a suburb of Seattle, Washington. They would handle

$$\begin{array}{r} 64 \\ -\ 28 \\ \hline \end{array}$$

by asking: "How much must I add to 28 to get 64?" They would keep track of this as follows:

$28 + 2 = 30$

so we write

$$\begin{array}{r} 64 \\ -\ 28 \\ \hline 2 \end{array}$$

$30 + 30 = 60$

so we write

$$\begin{array}{r} 64 \\ -\ 28 \\ \hline 2 \\ 30 \end{array}$$

$60 + 4 = 64$

so we write

$$\begin{array}{r} 64 \\ -\ 28 \\ \hline 2 \\ 30 \\ 4 \end{array}$$

Now--how much have we had to add to 28, in order to reach 64?

so we write

$$\begin{array}{r} 64 \\ -\ 28 \\ \hline 2 \\ 30 \\ 4 \\ \hline 36 \end{array} \; .$$

iv. The "Volkswagen" Algorithm. This algorithm was made up by a professor of mathematics; it uses "Volkswagen" numerals.

The problem

$$\begin{array}{r} 64 \\ -\ 28 \\ \hline \end{array}$$

would be handled by converting to Volkswagen numerals:

$$\begin{array}{r} 5_{14} \\ -\ 28 \\ \hline \end{array}$$

14 - 8 = 6

so we write

$$\begin{array}{r} 5_{14} \\ -\ 28 \\ \hline \end{array}$$

(Notice, of course, that the "Volkswagen" numeral

$$5_{14}$$

means: "5 tens and 14 ones." It is not the same as the standard decimal numeral

514,

which would mean: "5 hundreds, 1 ten, and 4 ones.")

Continuing,

50 − 20 = 30

so we write

$$\begin{array}{r} 5_{14} \\ -\,28 \\ \hline 36 \end{array}$$

Question #9: Can you subtract, using Volkswagen numerals, and then re-write the answer as a standard decimal numeral?

$$\begin{array}{r} 19_{21\;\;35} \\ -\;\;5_{19\;\;20} \\ \hline \end{array}$$

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SUPPLEMENTARY MODERN MATHEMATICS

For Grades 1-9

REFERENCE BOOK

Eleventh Session

3. Can you "find the sum" by two different methods?

a)
$$\begin{array}{r} 5\bar{3} \\ \underline{+\ 24} \end{array}$$

First Method: Using non-standard numerals.

$$^{-}3 + {}^{+}4 = {}^{+}1,$$

so we write

$$\begin{array}{r} 5\bar{3} \\ +\ 24 \\ \hline 1\,; \end{array}$$

and

$$50 + 20 = 70,$$

so we write

$$\begin{array}{r} 5\bar{3} \\ +\ 24 \\ \hline 1 \\ 70 \\ \hline 71\,. \end{array}$$

Second Method: Using Standard Numerals.

$$\begin{array}{r} 5\bar{3} \\ \\ \underline{+\ 24} \end{array}$$

$5\bar{3}$ means $50 + {}^{-}3$ or $50 - 3$, so we see that $5\bar{3}$ names the same number as $40 + 7$ or 47, and we write

$$\begin{array}{r} 47 \\ \underline{+\ 24} \end{array}$$

and handle it in the usual way.

b) Using non-standard numerals we get:

$$\begin{array}{r} 106\bar{6} \\ +\ 17\bar{3}2 \\ \hline \bar{4} \\ 30 \\ 700 \\ 2000 \\ \hline 273\bar{4} \end{array}$$

Using standard numerals we note that $106\bar{6}$ names the same number as 1054, and that $17\bar{3}2$ names the same number as 1672; so for $\begin{array}{r} 106\bar{6} \\ +\ 17\bar{3}2 \\ \hline \end{array}$ we write $\begin{array}{r} 1054 \\ +\ 1672 \\ \hline \end{array}$

and through the usual procedures we get 2726, which names the same number as $273\bar{4}$.

Agenda: Item 2. Non-Standard Numerals in General

Question #1.

Jean:	5 "super-bundles"	3 "bundles"	9 loose ones
Jacqueline:	1 "super-bundle"	4 "bundles"	7 loose ones
Combined:	6 "super-bundles"	7 "bundles"	16 loose ones

If consistent in practice we would take 10 of the 16 for a bundle, leaving six loose ones. We would then declare the combined total to be 6 "super-bundles," 8 "bundles," and 6 loose tongue depressors; i.e., 686.

Question #2.

$$\begin{array}{r} 192 \\ +\ 235 \\ \hline 427 \end{array}$$

In terms of "bundling," 192 represents 1 super-bundle, 9 bundles, and 2 loose ones; 235 represents 2 super-bundles, 3 bundles, and 5 loose ones. Upon addition we would find we had 3 super-bundles, 12 bundles, and 7 loose ones. If we then formed 1 super-bundle from 10 of the bundles we would have 4 super-bundles, 2 bundles, and 7 loose ones; i.e., 427.

i) Non-Decimal Numerals

Question #3.

Doris has 1 "super-bundle," 2 bundles, and 1 loose one. Roy has 2 bundles and 1 loose one. Together they have 1 "super-bundle," 4 "bundles," and 2 loose tongue depressors. Since the "super-bundles" are composed of three bundles in this situation, we take the three bundles for another "super-bundle" and describe the total as: 2 "super-bundles," 1 bundle, and 2 loose tongue depressors; i.e., 212_3 which is read "two one two, base three." The small numeral 3 to the right of and below 212 reminds us that this is a name for two "super-bundles" of 3 threes, and one bundle of three, and two loose ones, rather than two-hundred and twelve. To help avoid errors in our thinking, it is helpful to always read 212_3 as "two one two, base three," rather than two-hundred twelve base three, which is not what we mean at all.

When discussing expanded notation, why have we said $10^0 = 1$?

If $10^2 = 10 \times 10$,

and $10^3 = 10 \times 10 \times 10$,

and $10^7 = 10 \times 10 \times 10 \times 10 \times 10 \times 10 \times 10$,

then $10^a = 10 \times 10 \times 10 \times 10 \times 10 \ldots \times 10$, where 10 is a factor "a" times.

In multiplication of numbers involving exponents, we note that we can prove

$$10^3 \times 10^2 = 10^{(3+2)} = 10^5 \quad ,$$

since

$$10^3 \times 10^2 = (10 \times 10 \times 10) \times (10 \times 10) = (10 \times 10 \times 10 \times 10 \times 10) = 10^5$$

and

$$(10^a) \times (10^b) = 10^{a+b} ,$$

since

$$(10^a) \times (10^b) = \overset{\longleftarrow \text{ a times } \longrightarrow}{(10 \times 10 \times 10 \times \ldots 10)} \ \overset{\longleftarrow \text{ b times } \longrightarrow}{(10 \times 10 \times 10 \times 10 \times \ldots 10)}$$

$$= \overset{\longleftarrow \text{ a + b times } \longrightarrow}{(10 \times 10 \times 10 \times 10 \times 10 \times \ldots 10)} = 10^{a+b} .$$

Also,

$$\frac{10^5}{10^3} = 10^{5-3} = 10^2 ,$$

since

$$\frac{10^5}{10^3} = \frac{10 \times 10 \times 10 \times 10 \times 10}{10 \times 10 \times 10} = 10^2 .$$

To carry this line of thought further, we have

$$\frac{10^a}{10^b} = 10^{a-b} ,$$

and most specifically

$$\frac{10^a}{10^a} = 10^{a-a} = 10^0 .$$

But, $\frac{\square}{\square}$ = 1 for all numbers except zero; therefore, $\frac{10^a}{10^a} = 1$,

so we say

$$1 = \frac{10^a}{10^a} = 10^{a-a} = 10^0; \quad \text{i.e.,} \quad 1 = 10^0 .$$

iii) Roman Numerals

Question #4.

How many different names for numbers are represented in the box? Since only one name, 2, appears twice among those listed, we find ten names represented. 20_3 , $\frac{12}{2}$, 9 − 3, 5 + 1, VI , $1\bar{4}$, and 6 each name the same number. 2, II , 1 + 1, and 2 name the same number, therefore, only two numbers are represented.

20_3

$\frac{12}{2}$

(9 − 3)

2

II

5 + 1

1 + 1

VI

6

$1\bar{4}$

2

iv) "Volkswagen" Numerals

Question #5.

Which numeral names the larger number 1964 or $18_{16}4$?

$18_{16}4$ names the number 1 thousand, 8 hundreds, 16 tens, and 4 ones. But 1 thousand, 8 hundreds, 16 tens, and 4 ones = 1 thousand, 9 hundreds, 6 tens, and 4 ones, since 16 tens = 1 hundred plus 6 tens. Therefore, both numerals name the same number: 1 thousand, 9 hundreds, 6 tens, and 4 ones.

Question #6.

Using Volkswagen numerals the subtraction

$$\begin{array}{r} 12_{12}7 \\ -1196 \\ \hline 131 \end{array}$$

gives us a standard decimal numeral.

Agenda: Item 3. Non-Standard Algorithms

i. Kye's Algorithm for Subtraction

Question #7.

Subtract: $\begin{array}{r} 1964 \\ -\ 799 \\ \hline \end{array}$

$$4 - 9 = {}^{-}5$$

$$60 - 90 = {}^{-}30$$

$$900 - 700 = 200$$

$$1000 - 0 = 1000$$

so we write

$$\begin{array}{r} 1964 \\ -\ 799 \\ \hline {}^{-}5 \\ {}^{-}30 \\ 200 \\ 1000 \\ \hline 1165 \end{array},$$

or more simply

$$\begin{array}{r} 1964 \\ -\ 799 \\ \hline {}^{-}35 \\ 1200 \\ \hline 1165 \end{array}.$$

ii. Algorithm with Negative Digits

Question #8

Add by 3 different methods:

$$\begin{array}{r} 19\bar{6}\bar{5} \\ +\ 73\bar{4}5 \\ \hline \end{array}$$

Method 1: Use the negative digits.

$\bar{5} + 5 = 0$, so we write

$$\begin{array}{r} 19\bar{6}\bar{5} \\ +\ 73\bar{4}5 \\ \hline 0 \end{array};$$

and since $^{-}60 + {}^{-}40 = {}^{-}100$, we write

$$\begin{array}{r} 19\bar{6}\bar{5} \\ +\ 73\bar{4}5 \\ \hline 0 \\ {}^{-}100 \\ 9200 \\ \hline \end{array}$$

which is then ⟶ 9100 .

Method 2: Conversion to standard numerals.

$19\bar{6}\bar{5}$ names the same number as 1900 – 60 – 5, or 1835.

$73\bar{4}5$ names the same number as 7300 – 40 + 5, or 7265.

Therefore, for $\begin{array}{r} 19\bar{6}\bar{5} \\ +\ 73\bar{4}5 \\ \hline \end{array}$ we write $\begin{array}{r} 1835 \\ +\ 7265 \\ \hline \end{array}$,

and in the usual way we add, arriving at $\begin{array}{r} 1835 \\ +\ 7265 \\ \hline 9100 \end{array}$.

Method 3: "Balance Off"

$$\begin{array}{r} 19\bar{6}\bar{5} \\ +\ 73\bar{4}5 \\ \hline \end{array}$$

Since $3\bar{4}5$ names the same number as 265, we can think of our problem as equivalent to:

$$\begin{array}{r} 19\bar{6}\bar{5} \\ +\ 7265 \\ \hline \end{array};$$

but then $^{-}5 + 5 = 0$ and $^{-}60 + 60 = 0$, so we write

$$\begin{array}{r} 19\bar{6}\bar{5} \\ +\ 7265 \\ \hline 9100 \end{array}\ .$$

iv. The "Volkswagen" Algorithm.

Question #9: Subtract, using Volkswagen numerals:

$$\begin{array}{r} 19_{\;21\;\;35} \\ -\ \ 5_{\;19\;\;20} \\ \hline 14_{\;2\;\;15}\,; \end{array}$$

but this is $142_{\;15}\,,$

which is $143_{\;5}\,,$

which is $1435.$

SUPPLEMENTARY MODERN MATHEMATICS

For Grades 1 - 9

An In-Service Course for Teachers

TWELFTH SESSION

Review

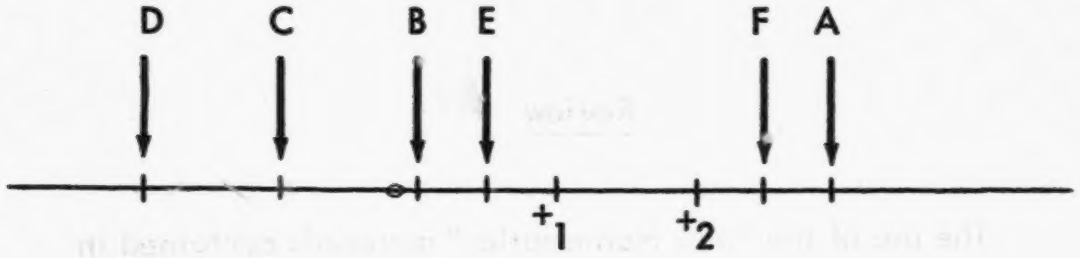
The use of the "new mathematics" materials contained in these various sessions depends upon many considerations, including some which are pedagogical, and some which are purely mathematical. Both are important. In the present session, we focus on a review of mathematical aspects of this material. The review takes the form of a sequence of questions, which you may wish to work individually, or to discuss as a group, or both.

I. Pebbles-in-the-bag

1. The use of "pebbles-in-the-bag" as an initial introduction of signed numbers can best be reviewed if someone will volunteer to teach this to the group, followed by suggestions and discussions.

II. The Number Line

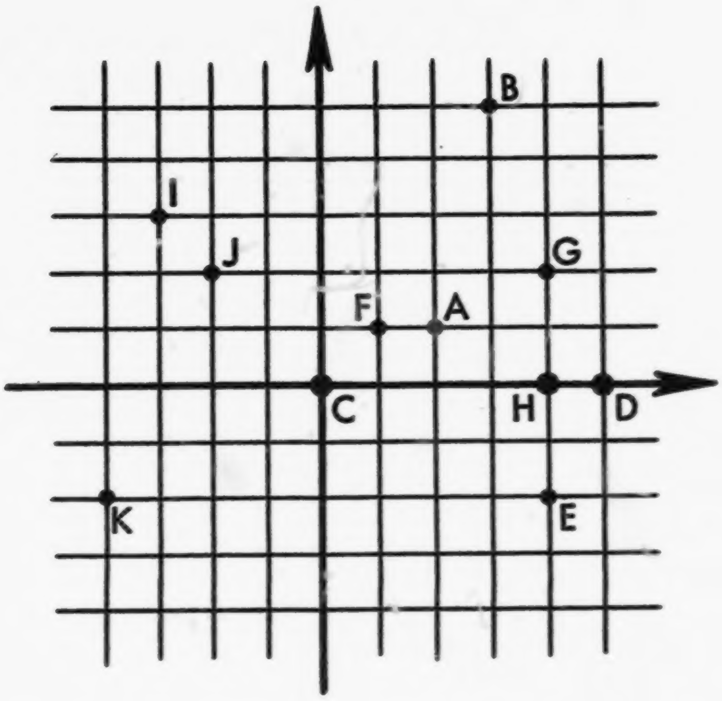
2. Can you attach proper number names to the points A, B, C, D, E, and F.



3. How could you use pieces of string, on a number line drawn on the blackboard, to indicate addition (for example, 2 + 3 = 5)? How could you use the string to locate $\frac{1}{2}$? $\frac{1}{3}$? $\frac{1}{4}$? $2\frac{1}{2}$?

III. Cartesian co-ordinates, and Tic Tac Toe

4. Can you match up the points on the graph, with the proper ordered pair of numbers?



$(^-4, ^-2)$

$(^-3, ^+2)$

$(^+2, ^-3)$

$(^-3, ^+3)$

$(^-2, ^+2)$

$(^+1, ^+1)$

$(^+5, 0)$

$(0, ^+5)$

$(^+4, 0)$

$(^+4, ^+2)$

$(^+4, ^-2)$

$(^+2, ^+1)$

$(^+3, ^+5)$

$(0, 0)$

5. You can best review Tic Tac Toe by playing a game yourself. We suggest that you use a 5 - by - 5 "board," and require 4 in an uninterrupted straight line, to win. Play it for all non-negative numbers, and then including negative numbers.

IV. "Equality," "True," "False," "Open"

6. What do mathematicians mean by the sign " = ," as in

   a) IV = 4

   b) 1 + 1 = 2

   c) 3 + 4 = 4 + 3?

   d) George Washington = the first President of the USA

7. What do mathematicians mean by a "variable"? Give an example.

8. Which are "true," which are "false," and which are "open"?

   a) $\frac{1}{2} + \frac{1}{3} = \frac{1}{5}$

   b) 3 + □ = 5

c) $53 + 37 = 80$

d) in base 2, $100 + 101 = 1001$

e) $2 \times 2\frac{1}{2} = 5$

f) ${}^{-}3 < {}^{-}1$

g) $3 < \square < 8$

h) $\square \times 0 = 0$

9. Can you state, carefully, the "rule for substituting," as it applies to

$$\square + \square = 12?$$

10. Can you state carefully, the "rule for substituting," as it applies to

$$\square + \triangle = 10?$$

11. Can you state the "rule for substituting," as it applies to

$$(\square + \triangle) + (\square + \square) = 12 + \triangle\ ?$$

12. Which of the following substitutions violate the "rule for substituting"?

a) $\boxed{10} + \boxed{10} = 20$

b) $\boxed{2} \times \boxed{4} = 8$

c) $\boxed{3} \times \boxed{3} = 8$

d) $\boxed{4} \times \triangle\!\!\!\!\!\!{\scriptstyle 2}\; = 8$

e) $\boxed{3} \times \triangle\!\!\!\!\!\!{\scriptstyle 2}\; = 10$

f) $\boxed{3} + \triangle\!\!\!\!\!\!{\scriptstyle 3}\; = 4$

g) $(\boxed{4} + \triangle\!\!\!\!\!\!{\scriptstyle 2}\;) + (\boxed{4} + \boxed{4}) = 12 + \triangle\!\!\!\!\!\!{\scriptstyle 2}$

h) $(\boxed{1} + \triangle\!\!\!\!\!\!{\scriptstyle 2}\;) + (\boxed{4} + \boxed{5}) = 12 + \triangle\!\!\!\!\!\!{\scriptstyle 2}$

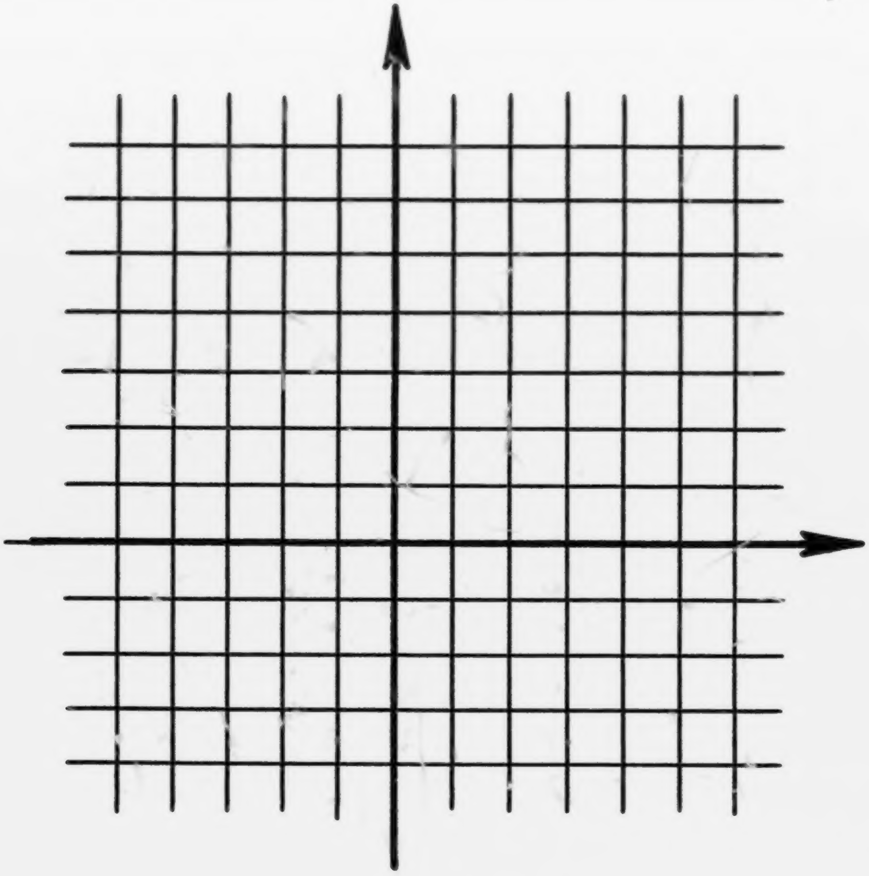
i) $(\boxed{5} + \triangle\!\!\!\!\!\!{\scriptstyle 1}\;) + (\boxed{5} + \boxed{5}) = 12 + \triangle\!\!\!\!\!\!{\scriptstyle 1}$

13. What do we mean by the "truth set" of an open sentence? Can you give an example?

V. Graphs of Functions

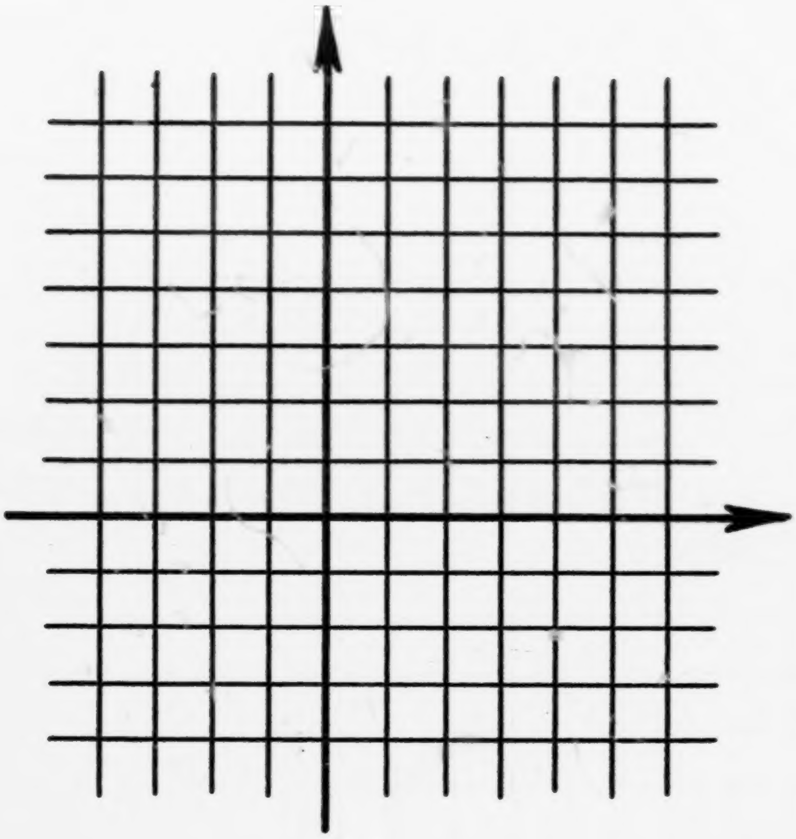
14. Using only integers, can you locate five points on the graph for the truth set of the open sentence

$$(1 \times \square) + 3 = \triangle ?$$



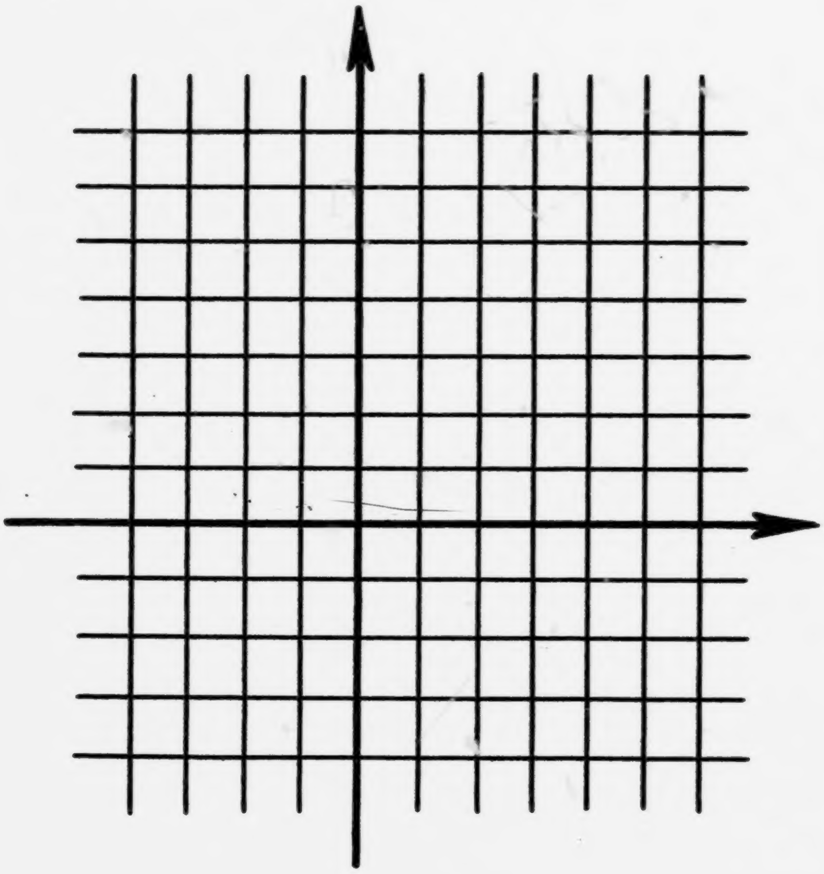
15. Using only integers, can you locate five points on the graph of the truth set for

$$(2 \times \square) + 3 = \triangle ?$$



16. Using only integers, can you locate five points on the graph for the equation

$$(2 \times \square) + 4 = \triangle ?$$



17. Can you describe the "slope" pattern, for the graph of an equation of the form

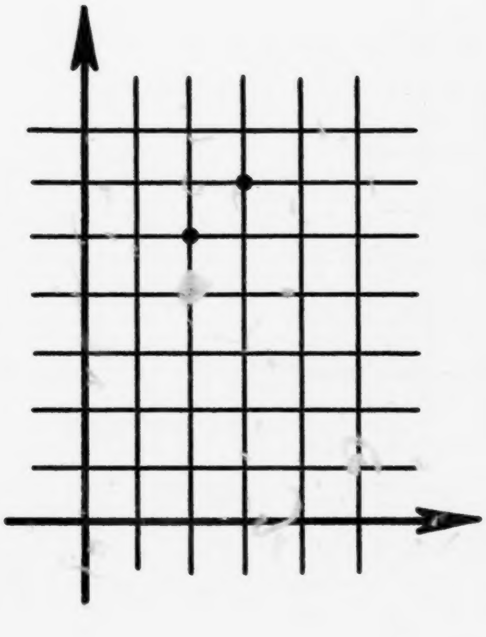
$$(a \times \square) + b = \triangle \,?$$

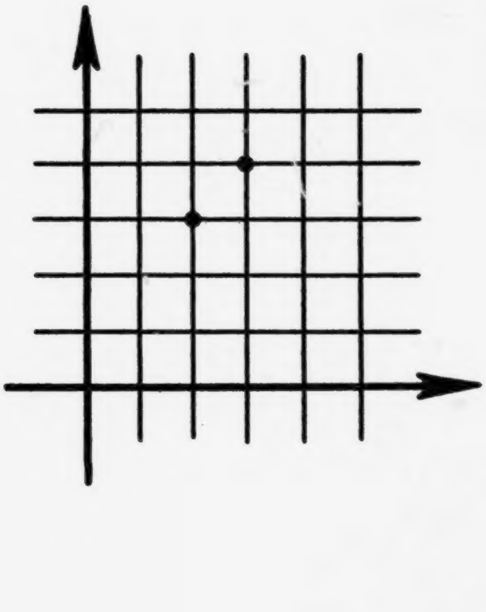
18. Can you describe the "intercept" pattern, for the graph of

$$(a \times \square) + b = \triangle \,?$$

Can you give examples?

19. On each graph, let us agree that the two points indicated determine a straight line. Write the equation of each line.





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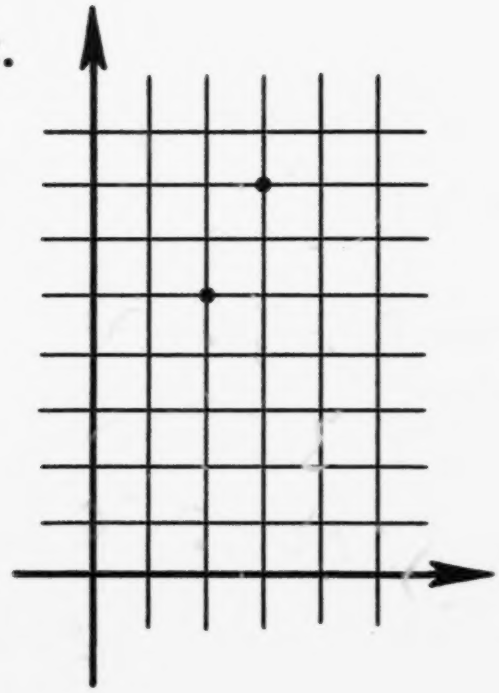

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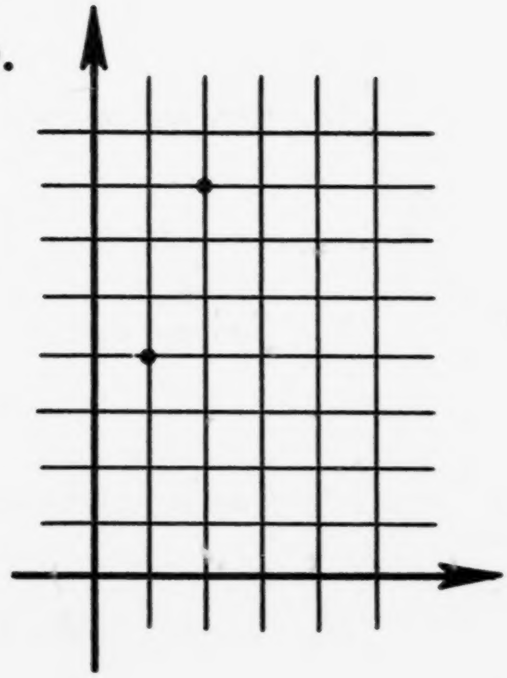

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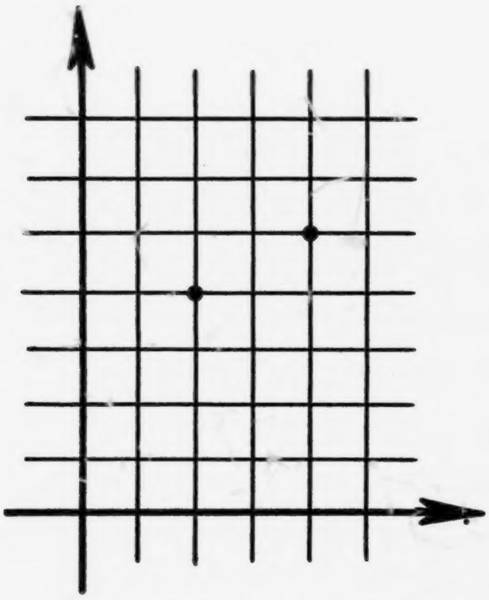

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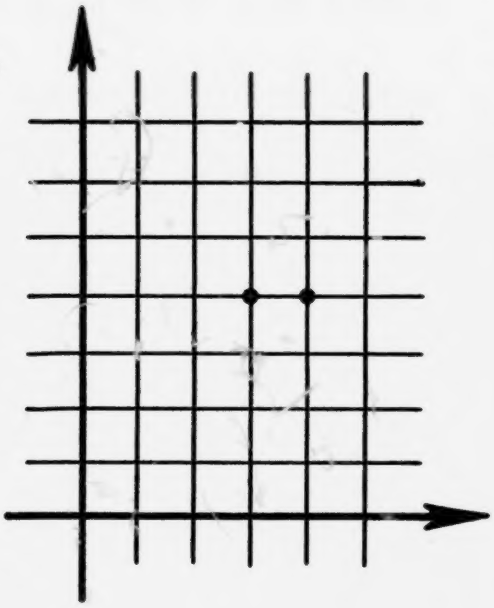
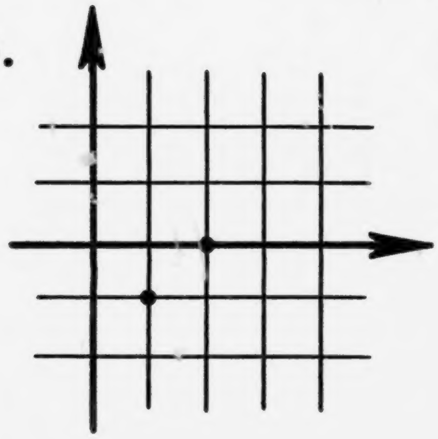
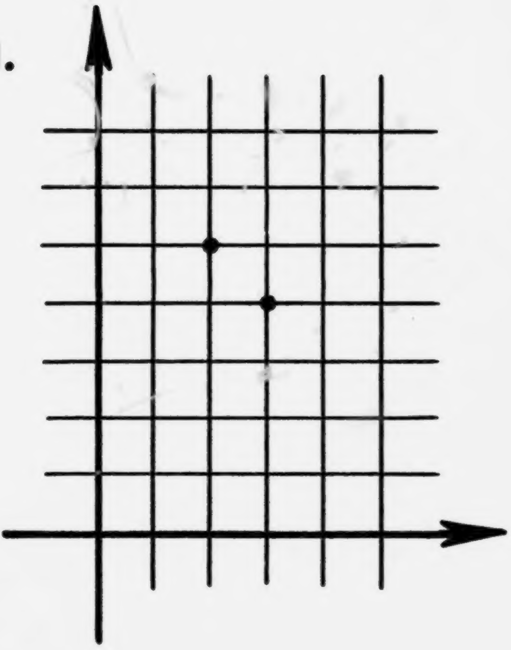


G.



H.



VI. Quadratic Equations

20. Can you find the truth set for each equation?

A. $(\square \times \square) - (5 \times \square) + 6 = 0$

B. $(\square \times \square) - (6 \times \square) + 5 = 0$

C. $(\square \times \square) - (8 \times \square) = 15 = 0$

D. $(\square \times \square) - (12 \times \square) + 20 = 0$

E. $(\square \times \square) - (9 \times \square) + 20 = 0$

F. $(\square \times \square) - (21 \times \square) + 20 = 0$

21. --A harder problem, for experts only!

Can you make a graph to show the truth set for

$$(\square \times \square) - (12 \times \square) + 20 = \triangle,$$

using only integers? Can you relate this to the truth set of the equation

$$(\square \times \square) - (12 \times \square) + 20 = 0\ ?$$

VII. Functions

22. Perhaps you can best review "functions" by playing the "make up a rule" game, as described in Sessions 5 and 6.

23. Do the formulas

$$(\square + 2) \times 3 = \triangle$$

and

$$(3 \times \square) + 6 = \triangle$$

represent the same function, or not?

24. Do the formulas

$$(\square + 3) \times 5 = \triangle$$

and

$$(\square \times 5) + 3 = \triangle$$

represent the same function, or not?

25. Can you give a "modern mathematics" definition of a "function"?

VIII. "Postman Stories" for Signed Numbers

26. What 4 things can the postman do? To what mathematical operation does each correspond?

27. Can you make up a "postman story" to correspond to each of the following:

a) ${}^{+}3 + {}^{+}7$

b) ${}^{-}2 + {}^{-}1$

c) ${}^{+}8 + {}^{-}2$

d) ${}^{+}10 + {}^{-}12$

e) ${}^{+}2 \times {}^{+}3$

f) ${}^{+}2 \times {}^{-}5$

g) ${}^{+}10 - {}^{+}2$

h) ${}^{-}10 - {}^{-}2$

i) ${}^{+}3 - {}^{-}1$

j) ${}^{-}5 - {}^{+}2$

k) ${}^{-}2 \times {}^{+}3$

l) ${}^{-}2 \times {}^{-}5$

m) ${}^-1 \times {}^-1$

n) ${}^-1 + {}^-1$

o) ${}^-1 - {}^-1$

28. Suppose that Mr. Jones has no confidence in "postman stories." Make a graph for the truth set of

$$({}^-1 \times \square) + {}^-3 = \triangle.$$

How could you use this graph to help convince Mr. Jones?

29. How can problems of "double counting" and similar confusions arise in using "postman stories"? How can such confusion be avoided?

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SUPPLEMENTARY MODERN MATHEMATICS

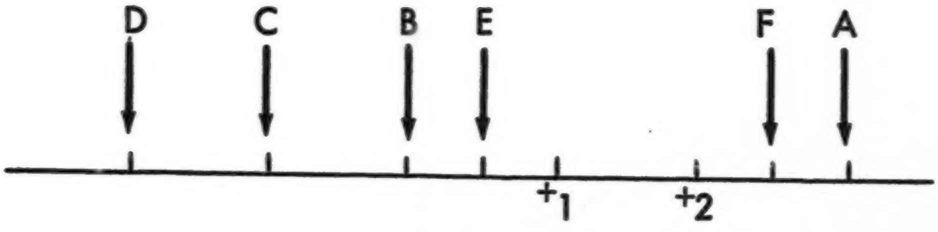
For Grades 1-9

REFERENCE BOOK

Twelfth Session

II. The Number Line

2. Attach proper number names to the points A, B, C, D, E, and F.



A $\longrightarrow$ $^{+}3$

B $\longrightarrow$ 0

C $\longrightarrow$ $^{-}1$

D $\longrightarrow$ $^{-}2$

E $\longrightarrow$ $^{+}\frac{1}{2}$

F $\longrightarrow$ $^{+}2\frac{1}{2}$

3. How could you use a piece of string on a number line drawn on the blackboard to indicate addition?

For addition of unit lengths, the string can be grasped forming a length equal to the distance from one to zero on the number line. The problem of addition is reduced to a repeated "laying off" of lengths. 2 + 3 is handled by laying off the length, "1," twice, and then three more times.

Even more basic is the fact that as soon as a child has the number line in front of him, he really has a model of a portion of the entire real number system. He is not restricted to the counting numbers, integers, or even to the rational numbers. He can indicate any point on the line and consider meaningfully the number represented by this length. As an example, notice the portion of the number line below. A child may indicate any point, (e.g. α), and with the help of his string, do any of the following and many more.

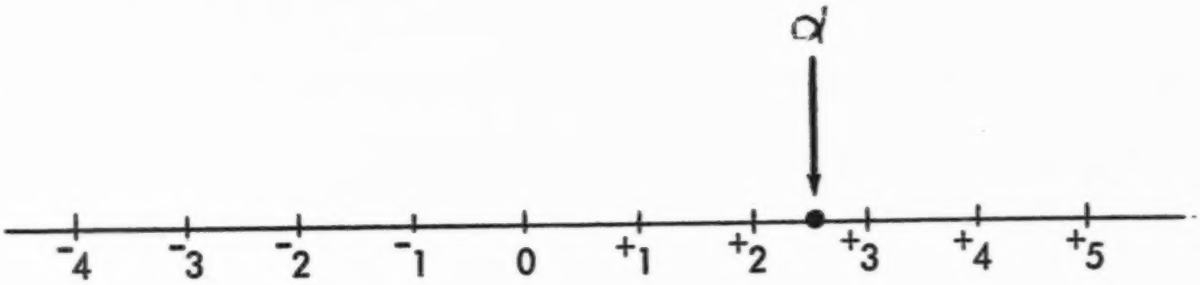
2α

$\alpha + 1$

$^-2 + \alpha$

$1 + \alpha$

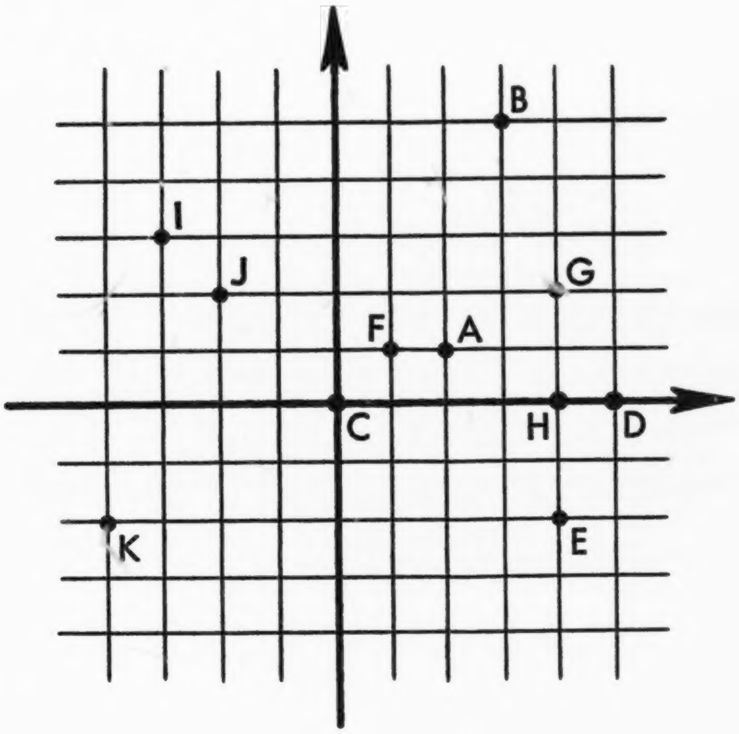
$\alpha - 3$



For additional insight into work with the number line, see "Goals for School Mathematics," the Report of the Cambridge Conference on School Mathematics, pp. 31-33.

III. Cartesian Coordinates, and Tic Tac Toe.

4. Match up the points on the graph with the proper ordered pairs of numbers.



$(^-4, ^-2)$

$(^-3, ^+2)$

$(^+2, ^-3)$

$(^-3, ^+3)$

$(^-2, ^+2)$

$(^+2, ^+1)$

$(^+3, ^+5)$

$(^+1, ^+1)$

$(^+5, 0)$

$(0, ^+5)$

$(^+4, 0)$

$(^+4, ^+2)$

$(^+4, ^-2)$

$(0, 0)$

A $\longrightarrow (^+2, ^+1)$

B $\longrightarrow (^+3, ^+5)$

C $\longrightarrow (0, 0)$

D $\longrightarrow (^+5, 0)$

E $\longrightarrow (^+4, ^-2)$

F $\longrightarrow (^+1, ^+1)$

G $\longrightarrow (^+4, ^+2)$

H $\longrightarrow (^+4, 0)$

I $\longrightarrow (^-3, ^+3)$

J $\longrightarrow (^-2, ^+2)$

K $\longrightarrow (^-4, ^-2)$

Notice that the ordered pairs ($^-3$, $^+2$), ($^+2$, $^-3$), and (0, $^+5$) do not name a lettered point on the graph given to us. They do name specific points, but not any of those points A through K.

IV. "Equality," "True," "False," "Open"

6. What do mathematicians mean by the sign " = ," as in

a) IV = 4

b) $1 + 1 = 2$

c) $3 + 4 = 4 + 3$

d) George Washington = the first President of the U.S.A.

The symbol " =," as used by mathematicians, could be translated to read: "Names the same number as," or "is another name for." $1 + 1 = 2$ says to the mathematician that $1 + 1$ is another name for 2.

7. What do mathematicians mean by a "variable"?

We have understood that when we speak of sentences in the mathematical sense, we are speaking of declarative sentences. If the subject of the sentence names a specific thing, we are able to assign a truth value (T or F) to the sentence, and we have called this kind of sentence a statement. A sentence not having a specific subject, thereby giving us a range of possible subjects from our universe of discourse, is said to exhibit a variable. This variable is commonly indicated by $\square$, $\triangle$, ∇ , x, y, etc. It serves as a reminder to us that we are confronted by a sentence representative of the whole list of statements possible upon substitution of each element of the replacement set.

8. Which are "true," which are "false," and which are "open?"

a)	$\frac{1}{2} + \frac{1}{3} = \frac{1}{5}$	false
b)	$3 + \square = 5$	open
c)	$53 + 37 = 80$	false
d)	in base 2, $100_2 + 101_2 = 1001_2$	true
e)	$2 \times 2\frac{1}{2} = 5$	true
f)	${}^-3 < {}^-1$	true
g)	$3 < \square < 8$	open
h)	$\square \times 0 = 0$	open

Statements like ${}^-3 < {}^-1$ sometimes give students trouble, so it seems a good idea to emphasize that $A < B$ means "A lies to the left of B on the number line."

In particular, $3 < \square < 8$ is an open sentence whose truth set consists of the numbers with the following restrictions. The numbers are to the left of 8 on the number line, and 3 is to the left of each of the numbers.

9. Can you state carefully the rule for substituting as it applies to $\square + \square = 12$?

In $\square + \square = 12$, the "rule for substituting" reminds us that we may use any element from the replacement set in $\square$ as long as we keep the intent in mind and use exactly the same element simultaneously in both $\square$'s.

10. In $\square + \triangle = 10$, the "rule for substituting" reminds us that we are not restricted here and may use any element of the replacement set for $\square$, and any element for $\triangle$.

11. $(\square + \triangle) + (\square + \square) = 12 + \triangle$.

The "rule for substituting" applies to the above sentence as follows: We may choose any element of the replacement set for substitution in $\square$, as long as we use the same element in every occurrence of $\square$. We may likewise choose any element for substitution in $\triangle$, as long as we use the same element in the second $\triangle$. This is the writer's intent.

12. Which of the following substitutions violate the "Rule for Substituting"?

a) $\boxed{10} + \boxed{10} = 20$

b) $\boxed{2} \times \boxed{4} = 8$

c) $\boxed{3} \times \boxed{3} = 8$

d) $\boxed{4} \times \triangle\!\!\!\!{\scriptstyle 2} = 8$

e) $\boxed{3} \times \triangle\!\!\!\!{\scriptstyle 7} = 10$

f) $\boxed{3} + \triangle\!\!\!\!{\scriptstyle 3} = 4$

g) $(\boxed{4} + \triangle\!\!\!\!{\scriptstyle 7}) + (\boxed{4} + \boxed{4}) = 12 + \triangle\!\!\!\!{\scriptstyle 7}$

h) $(\boxed{1} + \triangle\!\!\!\!{\scriptstyle 7}) + (\boxed{4} + \boxed{5}) = 12 + \triangle\!\!\!\!{\scriptstyle 7}$

i) $(\boxed{5} + \triangle\!\!\!\!{\scriptstyle 1}) + (\boxed{5} + \boxed{5}) = 12 + \triangle\!\!\!\!{\scriptstyle 1}$

Statements "b" and "h" display substitutions which violate the "rule for substituting." It is interesting to note that "b" is a true statement, while "h" is false. Truth value of a statement has no bearing on whether or not a "legal" substitution was made. In both "b" and "h" a substitution was made which did not reflect the intent of the statement.

13. What do we mean by the "truth set" of an open sentence?

The set of elements which gives us a true statement upon legal substitution into an open sentence is the truth set of the sentence.

The truth set of $\square + 2 = 5$ is $\{3\}$.

The truth set of $(\square \times \square) - (5 \times \square) + 6 = \triangle$ is $\{2,3\}$.

The truth set of □+ 2 = △ is indicated by this table, if we restrict our universe of discourse to the integers.

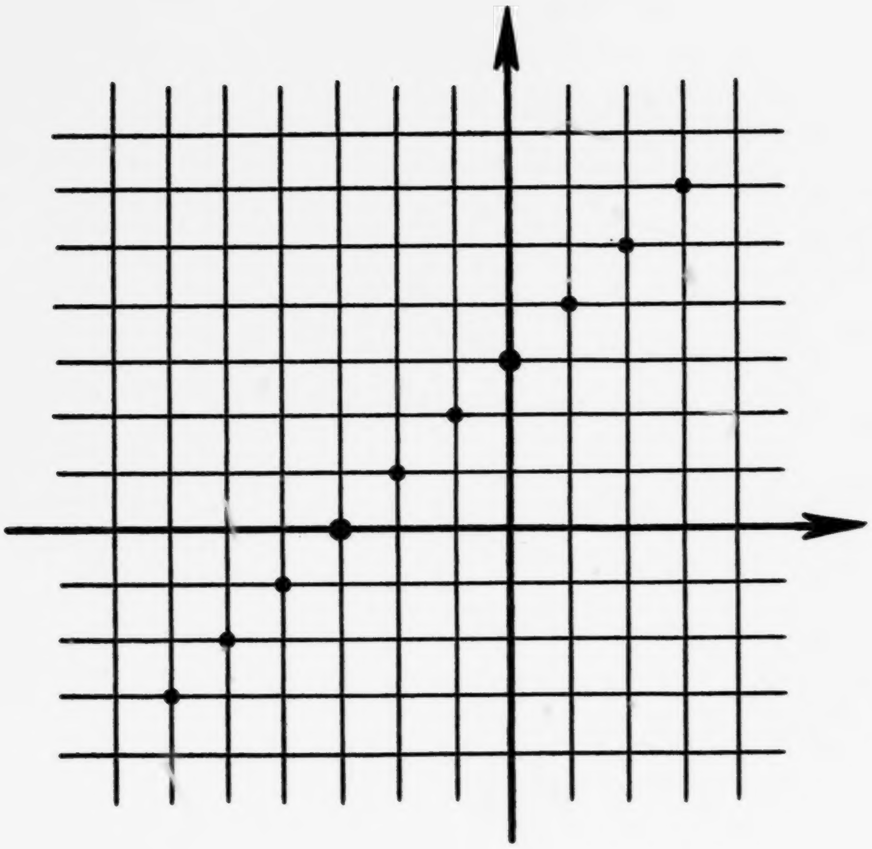
□	△
0	$^{+}2$
$^{+}1$	$^{+}3$
$^{-}1$	$^{+}1$
$^{+}2$	$^{+}4$
$^{-}2$	0

⋮

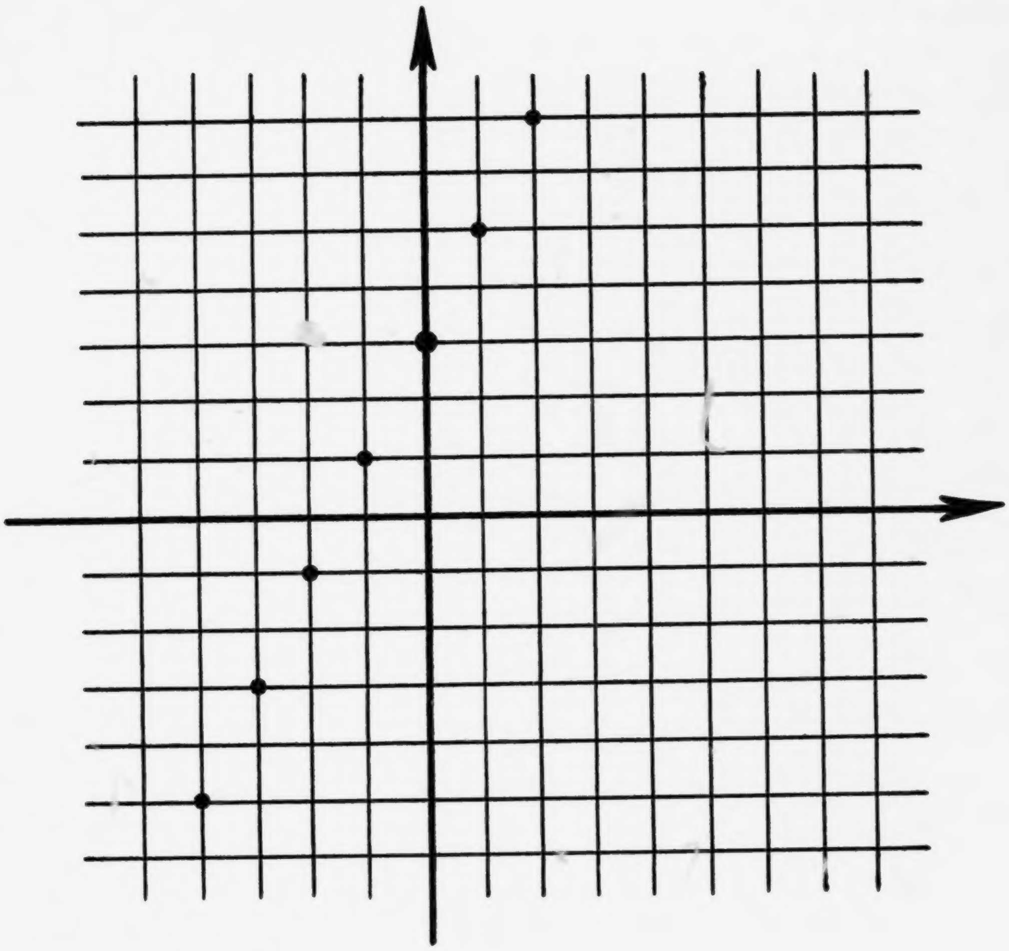
V. Graphs of functions.

14. Using only integers, locate five points in the truth set of the open sentence:

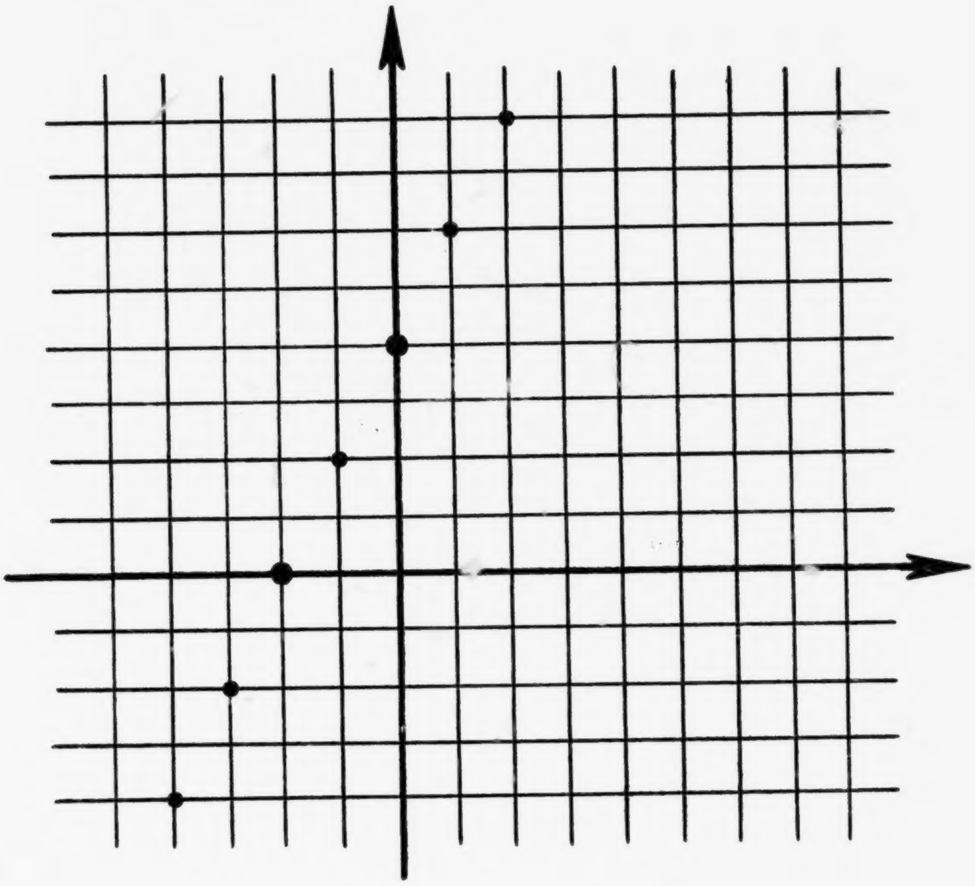
(1 × □) + 3 = △



15. $(2 \times \square) + 3 = \triangle$



16. $(2 \times \square) + 4 = \triangle$



17. Can you describe the "slope" pattern for the graph of an equation of the form $(a \times \square) + b = \triangle$?

The pattern may be described in several ways. One way is as follows: over one unit to the right and vertically "a" units for each successive point.

18. Can you describe the "intercept" pattern for the graph of $(a \times \square) + b = \triangle$?

The pattern is straightforward. The graph has an intercept on the vertical axis at "b." As an example for 17 and 18 above, consider the equation $(^{-}3 \times \square) + 2 = \triangle$.

You should find the slope pattern to be over one to the right and vertically $^{-}3$ (i.e., down 3), and the vertical intercept should be $^{+}2$.

19. Given the two points on the graphs, write the equation of the straight line determined by each.

A. $(1 \times \square) + 3 = \triangle$

B. $(1 \times \square) + 1 = \triangle$

C. $(2 \times \square) + 1 = \triangle$

D. $(3 \times \square) + 1 = \triangle$

E. $(\frac{1}{2} \times \square) + 3 = \triangle$

F. $(0 \times \square) + 4 = \triangle$ or $\triangle = 4$

G. $(1 \times \square) - 2 = \triangle$

H. $(^{-}1 \times \square) + 7 = \triangle$

VI. Quadratic Equations.

20. Find the truth set for each equation.

A. $(\square \times \square) - (5 \times \square) + 6 = 0$ $\{^{+}3, ^{+}2\}$

B.	$(\square \times \square) - (6 \times \square) + 5 = 0$	$\{^{+}5, ^{+}1\}$
C.	$(\square \times \square) - (8 \times \square) + 15 = 0$	$\{^{+}3, ^{+}5\}$
D.	$(\square \times \square) - (12 \times \square) + 20 = 0$	$\{^{+}10, ^{+}2\}$
E.	$(\square \times \square) - (9 \times \square) + 20 = 0$	$\{^{+}4, ^{+}5\}$
F.	$(\square \times \square) - (21 \times \square) + 20 = 0$	$\{^{+}20, ^{+}1\}$

21. Making a graph of the truth set of $(\square \times \square) - (12 \times \square) + 20 = \triangle$.

The first job to consider is that of collecting ordered pairs of values for the truth set of this open sentence. We can then plot points using those coordinates. Below is a table of some of the points you might find.

$\square$	$^{-}1$	0	$^{+}1$	$^{+}2$	$^{+}3$	$^{+}4$	$^{+}5$	$^{+}6$	$^{+}7$	$^{+}8$	$^{+}9$	$^{+}10$	$^{+}11$	$^{+}12$ . . .
$\triangle$	33	20	9	0	$^{-}7$	$^{-}12$	$^{-}15$	$^{-}16$	$^{-}15$	$^{-}12$	$^{-}7$	0	9	20 . . .

The equation $(\square \times \square) - (12 \times \square) + 20 = 0$ has much the same appearance as $(\square \times \square) - (12 \times \square) + 20 = \triangle$. But, while the truth set of $(\square \times \square) - (12 \times \square) + 20 = \triangle$ was an unending set of ordered pairs of numbers, the truth set of $(\square \times \square) - (12 \times \square) + 20 = 0$ has just two single number elements: $\{10, 2\}$. What relation to the graph is there for these two numbers? Plot them to see.

VII. Functions

23. Do the formulas $(\square + 2) \times 3 = \triangle$ and $(3 \times \square) + 6 = \triangle$ represent the same function or not?

They do if and only if they have identical truth sets. You may want to find a partial table of truth values for each, or even graph both.

24. The formulas $(\square + 3) \times 5 = \triangle$ and $(\square \times 5) + 3 = \triangle$ may be subjected to the same tests.

$(\square + 3) \times 5 = \triangle$

$\square$	$\triangle$
$^{-}1$	$^{+}10$
0	$^{+}15$
$^{+}1$	$^{+}20$
$^{+}2$	$^{+}25$

$(\square \times 5) + 3 = \triangle$

$\square$	$\triangle$
$^{-}1$	$^{-}2$
0	$^{+}3$
$^{+}1$	$^{+}8$
$^{+}2$	$^{+}13$

25. A "modern" mathematics definition of function.

Earlier in our study, in the Sixth Session, we found that a function was appropriately described as a set of ordered pairs of numbers. After a good deal of experience with functions, we will want to be even more specific in our definition.

VIII. Postman Stories for Signed Numbers.

26. What 4 things can a postman do?

1. bring a check:	$+^{+}$	corresponds to addition to the "mental bank balance"
2. bring a bill:	$+^{-}$	corresponds to subtraction from the "mental bank balance"
3. take away a check:	$-^{+}$	corresponds to subtraction from the "mental bank balance"
4. take away a bill:	$-^{-}$	corresponds to addition to the "mental bank balance"

27. Making up "Postman Stories"

In making up "postman stories" remember that two distinct interpretations have been used.

When we are adding or subtracting signed numbers, we use one signed number for each bill or check. A delivery of a \$3 check and the pickup of a \$1 bill is represented as:

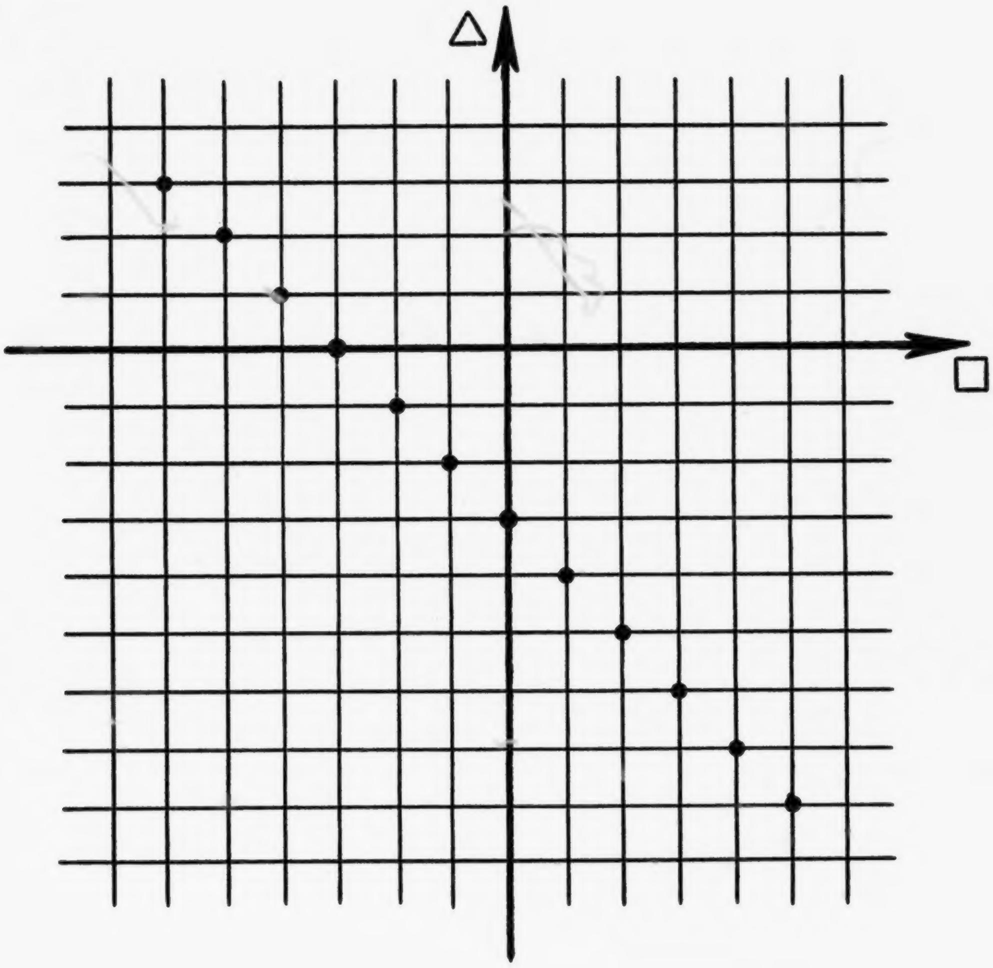
$$^{+}3 - {^{-}1} = {^{+}4}$$

When we want to multiply signed numbers, we specify that the first signed number determines the number of bills or checks handled in the visit, and the second signifies the size of each; e.g., a delivery of 3 bills for \$8 each is represented as:

$$^{+}3 \times {^{-}8} = {^{-}24}$$

28. Suppose Mr. Jones has no confidence in "postman stories."

$({^{-}1} \times \square) + {^{-}3} = \triangle$ has the graph below for its truth set.



How could you use this graph to help convince Mr. Jones?

One way might be as follows: since the graph represents quite concretely the relationship between the two variables, □ and △ , Mr. Jones will probably feel better with a method that seems to agree with this "rock solid" picture. You might ask him to consider the following type of comparison.

The postman story in which the postman picks up 1 check for \$4 and brings a bill for \$3 is represented in our postman model like this: $(^{-}1 \times 4) + {}^{-}3 = ?$. Our model persuades us that the housewife will be \$7 poorer and we represent this as $^{-}7$. As a comparison in $(^{-}1 \times \square) + {}^{-}3 = \triangle$, ask Mr. Jones to notice what number is in the △ when $^{+}4$ is in the □ . We really want him to notice that we count out $^{+}4$ on the □ axis, and down 7 to get to the point on the graph.

29. How can problems of "double counting" and similar confusion arise in using "postman stories"? How can such confusion be avoided?

As we become more proficient with the "postman stories," we may want to experiment a little ourselves. To keep a legitimate train of thought we must avoid any model which could introduce double counting; i.e., avoid any model which would affect our "mental checkbook balance" both on receipt and payment of a bill.